p-dimension of henselian fields

An application of Ofer Gabber's algebraization technique

Fabrice Orgogozo

CNRS, École polytechnique (France)

El Escorial, 2006-9-4

Definition (p-cohomological dimension)

1. Let X be a scheme and p a prime number. We say that $\mathrm{cd}_p(X) \leq N$ iff for all p-torsion étale sheaf $\mathscr{F}$ and all integer $i > N$, we have:

$$\mathrm{H}^i(X_{\acute{e}t}, \mathscr{F}) = 0.$$

Alexandre Grothendieck et al., SGA 4, exposé X by Michael Artin

2. Let G be a profinite group. We say that $\mathrm{cd}_p(G) \leq N$ iff for all discrete p-torsion G-module M (with continuous action) we have, for all $i > N$, we have:

$$\mathrm{H}^i(G, M) = 0.$$

Jean-Pierre Serre, Cohomologie galoisienne.

- If k is a field, $(\mathrm{Spec}(k))_{\acute{e}t} = \mathrm{B}G_k$ (where $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$), so

$$\mathrm{cd}_p(\mathrm{Spec}\, k) = \mathrm{cd}_p(G_k).$$

- Si X is an affine scheme of characteristic $p > 0$,

$$\mathrm{cd}_p(X) \leq 1.$$

It comes from the Artin-Schreier exact sequence

$$0 \to \mathbf{Z}/p \to \mathbf{G}_a \xrightarrow{\wp} \mathbf{G}_a \to 0.$$

Examples

Reminder: a C_1-field has p-cohomological dimension (for all prime number p) ≤ 1.

Theorem

1. *A finite field is C_1. $\widehat{\mathbf{Z}} \overset{\mathrm{Frob}_p}{\xrightarrow{\sim}} G_{\mathbf{F}_p}$ so $\mathrm{cd}_\ell\, \mathbf{F}_p$ is exactly 1, for all prime number ℓ.*
2. *If k is algebraically closed, $k(t)$ is C_1 (Tsen).*
3. *Let A be an henselian, excellent dvr with algebraically closed residue field. Then $\mathrm{Frac}(A)$ is C_1 (Lang).*

Excellent: the extension $\mathrm{Frac}(\widehat{A})/\mathrm{Frac}(A)$ is separable.

Corollary (of Tsen's result)

Let K/k a field extension of transcendence degree N and p a prime number. Then,

$$\mathrm{cd}_p(K) \leq N + \mathrm{cd}_p(k).$$

This is an equality if K/k is of finite type, $\mathrm{cd}_p(k) < +\infty$ and $p \cdot 1 \in k^{\times}$.

Corollary (of Lang's result)

Let K be a complete discrete valuation field with perfect residue field k and p a prime number. Then, we have:

$$\mathrm{cd}_p(K) \leq 1 + \mathrm{cd}_p(k).$$

This is an equality if $p \cdot 1 \in K^{\times}$.

Application: $\mathrm{cd}_p(\mathbf{Q}_p) = 2$.

If the residue field k is not perfect, Ω^1_k should be taken into account.

$$\mathrm{H}^\star(K, \mathbf{Z}/p(\star)) \overset{\text{Bloch-Katō}}{\longleftrightarrow} K^M_\star(K)/p,$$

via the *cohomological symbol* ($\leftarrow$).

$$K^M_\star(K) \longrightarrow \Omega^\star_K,$$

via the *differential symbol* ($\rightarrow$):

$$\{x_1, \ldots, x_r\} \mapsto \mathrm{dlog}(x_1) \wedge \cdots \wedge \mathrm{dlog}(x_r).$$

Theorem (Kazuya Katō (simplified version))

Let A be a henselian excellent discrete valuation ring of mixed characteristic $(0, p)$. Let K, k the corresponding fields. Then:

$$\mathrm{cd}_p(K) = 1 + \dim_p(k),$$

where $\dim_p(k)$ is equal to the p-rank of k, $\dim_k \Omega^1_k (= [k : k^p])$, or $\dim_k \Omega^1_k + 1$.

Definition (of the p-dimension $\dim_p$; first part)

Let κ be a field of characteristic $p > 0$ and $n \in \mathbf{N}$. We define $\mathrm{H}_p^{n+1}(\kappa)$ as the cokernel of the map (also denoted by "$1 - C^{-1}$"):

$$\Omega_\kappa^n \xrightarrow{\wp} \Omega_\kappa^n / d\Omega_\kappa^{n-1} \ : \ a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} \mapsto (a - a^p)\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n},$$

where

$$\Omega_\kappa^i := \bigwedge^i \Omega^1_{\kappa/\mathbf{Z}}$$

and

$$a \in \kappa,\ b_i \in \kappa^\times.$$

Characteristic p analogue of $\mathrm{H}^{n+1}(\mathrm{Spec}(\kappa)_{\acute{e}t}, \mu_p^{\otimes n})$.

$\mathrm{H}_p^1(\kappa) = \kappa/\wp(\kappa)$. Non zero for $\mathbf{F}_p$.
$\mathrm{H}_p^2(\kappa) = \mathrm{Br}(\kappa)[p]$.

Definition (of the p-dimension $\dim_p$; final part)

Let κ be a field, p a prime number.

1. Assume $\mathrm{char.}(\kappa) \neq p$. Then $\dim_p(\kappa) := \mathrm{cd}_p(\kappa)$.
2. Assume $\mathrm{char.}(\kappa) = p$.

$$\dim_p(\kappa) \leq N$$

iff

$$[\kappa : \kappa^p] \leq p^N \ \& \ \mathrm{H}_p^{N+1}(\kappa') = 0 \ \forall \, \kappa'/\kappa \text{ finite}$$

$\dim_p(\mathbf{F}_p) = 1$.

Remarks

The p-rank is invariant under finite field extension.
Need to consider $\kappa'/\kappa \longleftrightarrow \mathrm{H}_p^{r+1}$ is a "constant" coefficient cohomology theory.
(Cf. $\mathrm{R}\Gamma(G_\kappa, M)$ (for various p-torsion G_κ-modules M) $\longleftrightarrow \mathrm{R}\Gamma(G_{\kappa'}, \mathbf{Z}/p)$ (for finite étale κ'/κ).)

Theorem (K. Katō (final version); analogue of Lang's theorem)

Let A be a henselian excellent discrete valuation ring and p a prime number. Then,

$$\dim_p(K) = 1 + \dim_p(k).$$

Corollary (Analogue of Tsen's theorem)

Let K/k a field extension of transcendence degree N and p a prime number. Then,

$$\dim_p(K) \leq N + \dim_p(k).$$

Proof: use the "classical" formula for cd_p and the possibility to make K/k a "residue field extension" of a characteristic zero dvr extension.

Kazuya Katō.
Galois cohomology of complete discrete valuation rings.
LNM 967, 1980.

K. Katō's conjecture

Let A be an integral henselian, excellent (e.g. complete) local ring. Let K be its fraction field and k its residue field of characteristic $p > 0$. Then:

$$\dim_p(K) = \dim(A) + \dim_p(k).$$

(Here, $\dim(A)$ is the Krull dimension.)

Theorem (K. Katō, 1986)

Let A be a normal excellent henselian local ring of dimension 2 with residue field k and fraction field K. Suppose that k is algebraically closed. Then, for all prime number $p \neq \mathrm{char.}(K)$, we have:

$$\mathrm{cd}_p(K) = 2.$$

Remarks

- The proof uses the theorem of Merkur'ev-Suslin and resolution of singularities for surfaces.

  Shūji Saitō.
  Arithmetic on two dimensional local rings.
  Inventiones mathematicæ 85, 1986.

- This has been extended to an arbitrary residue field by Takako Kuzumaki.

In the following, we will K. Katō's conjecture, namely:

Theorem

Let A be an integral henselian, excellent local ring. Let K be its fraction field and k its residue field of characteristic $p > 0$. Then:

$$\dim_p(K) = \dim(A) + \dim_p(k).$$

Remark

The equal-characteristic formula is proved first and used to show the mixed-characteristic formula.

Lower bound: $\dim_p(K) \geq \dim(A) + \dim_p(k)$

Reduction to the normal case

We may assume A normal:

- A^ν / A is finite (A is excellent).
- $\dim_p$ is invariant by finite extension (when it is finite).
  The characteristic p case can be shown by using the classical result and the theorem of K. Katō or, more simply, by using the existence of trace maps on H_p^{r+1}.

Lower bound: $\dim_p(K) \geq \dim(A) + \dim_p(k)$

Induction using K. Katō's theorem (i.e. dim 1 case)

Let $\mathfrak{p}$ be a height one prime ideal
$L := \mathrm{Frac}\, A_{\mathfrak{p}}$, $B = \widehat{A_{\mathfrak{p}}}$ (complete dvr) and $\widehat{L} := \mathrm{Frac}\, B$.

Mixed characteristic: $G_{\widehat{L}} \hookrightarrow G_L$

$$\Rightarrow \mathrm{cd}_p(K = L) \geq \mathrm{cd}_p(\widehat{L}) \geq 1 + \dim_p(\mathrm{Frac}\, A/\mathfrak{p}) \quad \text{[K. Katō]}$$
$$\geq 1 + \big(\dim(A) - 1 + \dim_p(k)\big) \quad \text{[induction]}.$$

Equal characteristic:
$[L : L^p] \geq [\widehat{L} : \widehat{L}^p]$ (if $[L : L^p]$ is finite) and $\mathrm{H}^{r+1}_p(L) \twoheadrightarrow \mathrm{H}^{r+1}_p(\widehat{L})$.

Upper bound: $\dim_p(K) \leq \dim(A) + \dim_p(k)$
Ofer Gabber's algebraization technique

- Reduction to the complete case (Artin-Popescu; cf. excellency hyp.) $\Rightarrow$ finite over "good" ring (i.e. ring of power series).
- (Proof of) Nagata's Jacobian criterion (equal characteristic) $\Rightarrow$ generically étale (and finite) over good ring.
  Mixed characteristic: use Helmut Epp's result.
- Ramification locus finite over lower dimensional base (Weierstraß) and Renée Elkik's algebraization.

$\Rightarrow$ relative dimension 1

Upper bound: $\dim_p(K) \leq \dim(A) + \dim_p(k)$
Ofer Gabber's algebraization technique

Ofer Gabber.
A finiteness theorem for non abelian H^1 of excellent schemes.
Conférence en l'honneur de Luc Illusie, Orsay, 2005-6-27.

Ofer Gabber.
Finiteness theorems for étale cohomology of excellent schemes.
Conference in honor of Pierre Deligne, Princeton, 2005-10-17.

Michael Artin.
Cohomologie des préschémas excellents d'égales caractéristiques.
SGA 4, exposé XIX.

Upper bound: $\dim_p(K) \leq \dim(A) + \dim_p(k)$

Lemma

Let A be a local henselian quasi-excellent integral ring, $\widehat{A}$ its completion and $K, \widehat{K}$ the respective fraction fields. Then the K-algebra $\widehat{K}$ is a filtered colimit of K-algebras of finite type with retraction.

Definition

A ring A is quasi-excellent if it is noetherian and

- for all $x \in X = \mathrm{Spec}(A)$, the morphism $\mathrm{Spec}(\widehat{\mathscr{O}_{X,x}}) \to \mathrm{Spec}(\mathscr{O}_{X,x})$ is regular,
- for all A'/A finite, $\mathrm{Reg}(\mathrm{Spec}(A'))$ is open.

Such a ring is in particular "universally Japanese". For henselian local rings, "excellent"=quasi-excellent.

Proof.

Immediate corollary of Sorin Popescu's version of M. Artin's approximation theorem.

Theorem (S. Popescu; Artin's approximation property)

> *Any finite system of polynomial equations over A has a A-point iff it has a $\widehat{A}$-point.*

Corollary

Let A as above and F finite presentation functor $\left(A - \mathrm{Alg}\right) \to$ Set. Then $\mathrm{F}(K) \to \mathrm{F}(\widehat{K})$ is an injection.

Theorem (O. Gabber, conf. L. Illusie, lemma 8.1)

Let A be an integral local complete noetherian ring of dimension d with residue field k. There exists a subring A_0 of A, isomorphic to $k[[t_1, \ldots, t_d]]$ such that $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ is finite, generically étale.

Remark

This theorem is obvious in mixed characteristic (hence "generically of characteristic zero"). However, in the algebraization process, it is also used (see below).

Equicharacteristic upper bound: $\dim_p(K) \leq \dim(A) + \dim_p(k)$

$X = \mathrm{Spec}(A) \to X_0 = \mathrm{Spec}(A_0)$ is finite, generically étale. $X_0 \simeq \mathbf{A}^{d\wedge}_{k(o)}$.

Let $R \subset X_0$ ramification locus. WMA: point $(t_1, \ldots, t_{d-1}) \notin R \Rightarrow$ so (Weierstraß) $R \subset V(r)$, r monic polynomial in $k[[x_1, \ldots, x_{d-1}]][x_d]$.

Reminder on Weierstraß theorem

Theorem (Weierstraß, circa 1880)

1. *Let κ be a local complete ring (e.g. a field) with maximal ideal $\mathfrak{m}$, and $f \in \kappa[[t_1, \ldots, t_n]]$ with $f \equiv (u \in \kappa[[t_n]]^\times) \cdot t_n^N$ mod. $(\mathfrak{m}, t_1, \ldots, t_{n-1})$. Then $f = unit \cdot P$, where $P \in \kappa[[t_1, \ldots, t_{n-1}]][t_n]$.*
2. *For each element $f \in \kappa[[t_1, \ldots, t_n]]$ non zero mod $\mathfrak{m}$, there exists a κ-linear automorphism α, defined by $\alpha(t_i) = t_i + t_n^{c_i}$ for $i = 1, \ldots, n-1$ (and suitable c_i's), and $\alpha(t_n) = t_n$ such that $\alpha(f)$ is as in (1).*

$X = \mathrm{Spec}(A) \to X_0 = \mathrm{Spec}(A_0)$ is finite, generically étale. $X_0 \simeq \mathbf{A}^{d\wedge}_{k(o)}$.

Let $R \subset X_0$ ramification locus. WMA: point $(x_1, \ldots, x_{d-1}) \notin R \Rightarrow \Rightarrow$ so (Weierstraß) $R \subset V(r)$, r monic polynomial in $k[[x_1, \ldots, x_{d-1}]][x_d]$.

In particular:

- $V(r)$ comes from $\widetilde{X_0} := \mathrm{Spec}(k[[x_1, \ldots, x_{d-1}]]\{x_d\})$.
- The r-adic completion of $\widetilde{X_0}$ is X_0.
- The ("algebraized") pair $(\widetilde{X_0}, V(r))$ is henselian, $\Rightarrow$ we can use R. Elkik's theorem to descend $X \to X_0$ to $\widetilde{X} \to \widetilde{X_0}$.

Reminder on Renée Elkik's theorem

Definition

A pair $(X = \mathrm{Spec}(A), Y = V(I))$ is henselian if for every polynomial $f \in A[T]$, every simple root of f in A/I lifts to a root in A.

Theorem (Renée Elkik, 1973)

Let $(X = \mathrm{Spec}(A), Y = V(I))$ be an henselian pair with A noetherian. Let $\widehat{X}$ be the completion of X along Y and $\widehat{Y}$ be the corresponding closed subscheme. Assume for simplicity that the complement U of Y in X is connected. Then $\widehat{U} := \widehat{X} - \widehat{Y}$ is also connected and the map

$$\pi_1(U) \to \pi_1(\widehat{U})$$

is an isomorphism.

Renée Elkik
Solutions d'équations à coefficients dans un anneau hensélien.
Annales scientifiques de l'École normale supérieure, 1973.

Equicharacteristic upper bound: proof

Set $d = \dim(A)$, $r = \dim_k \Omega^1_k$, $n = r + d$.

$$\big(\dim_p K = \dim_K \Omega^1_K + \{0,1\}\big) \overset{?}{\leq} d + \big(\dim_p(k) = r + \{0,1\}\big).$$

$\dim_K \Omega^1_K \overset{\text{easy}}{=} \dim(A) + \dim_k \Omega^1_k \, (= d + r)$
$1 \overset{?!}{\leq} 0.$ $\longleftrightarrow$ $\mathrm{H}^{r+1}_p(k) = 0 \Rightarrow \mathrm{H}^{n+1}_p(K) = 0$ (applied to K'/K).

Reminder

If $\dim_\kappa \Omega^1_\kappa = r$, $\dim_p(\kappa) = r$, iff $\forall \kappa'/\kappa$ finite, $\mathrm{H}^{r+1}_p(\kappa') = 0$
where $\mathrm{H}^{r+1}_p(\kappa)$ is the cokernel of the map:

$$\Omega^r_\kappa \xrightarrow{\wp} \Omega^r_\kappa / d\Omega^{r-1}_\kappa \ : \ a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_r}{b_r} \mapsto (a - a^p)\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_r}{b_r}.$$

Assume $\mathrm{H}^{r+1}_p(k) = 0$ (and similarly for finite extension), and take an element in Ω^n_K.
Want to show it belongs to the image of $\wp$ (modulo exact forms).

Equicharacteristic upper bound: proof
2/3

As above: $d = \dim(A)$, $r = \dim_k \Omega^1_k$, $n = r + d$, assume $\mathrm{H}^{r+1}_p(k) = 0$, and consider $\frac{\omega}{f} \in \Omega^n_K$, where $\omega \in \Omega^n_A/\text{torsion}$ and $f \in \mathfrak{m}_A - \{0\}$.

Let $A_0 = k[[x_1, \ldots, x_d]]$ as in O. Gabber's theorem and we have by algebraization a cartesian diagram

$$\begin{array}{ccc} X = \mathrm{Spec}(A) & \longrightarrow & \widetilde{X} = \mathrm{Spec}(\widetilde{A}) \\ \downarrow & \square & \downarrow \text{finite, gen. étale} \\ X_0 = \mathrm{Spec}(A_0) & \longrightarrow & \widetilde{X_0} = \mathrm{Spec}(\widetilde{A_0}) \end{array}$$

where $\widetilde{A_0} = k[[x_1, \ldots, x_{d-1}]]\{x_d\}$.

We may assume $f \in A_0$ (by taking norms).

Equicharacteristic upper bound: proof
3/3

Up to a change of the first coordinates (Weierstraß), we may assume $f \in k[[x_1, \dots, x_{d-1}]][x_d]$ and is monic. (In particular, it belongs to $\widetilde{A_0} = k[[x_1, \dots, x_{d-1}]]\{x_d\}$.)
It follows, that A is the f-adic completion of $\widetilde{A}$.

Hence,

$$\frac{\omega}{f} = \left(\frac{\tilde{\omega}}{f} \in \Omega^n_{\widetilde{A}}/\text{tors.}\right) + \left(? \in \mathfrak{m}_A \Omega^n_A/\text{tors.}\right)$$

Surjectivity of $\wp$?

- Second term: easy. Cf. $a = \wp(a + a^p + a^{p^2} + \cdots)$.
- First term: $\operatorname{Frac} \widetilde{A}$ of transcendence degree one over $\operatorname{Frac} k[[x_1, \dots, x_{d-1}]] \Rightarrow$ use K. Katō's theorem and induction.

Mixed-characteristic upper bound: $\mathrm{cd}_p(K) \leq \dim(A) + \dim_p(k)$

Easier: $X = \mathrm{Spec}(A) \to X_0 = \mathrm{Spec}(A_0)$ ($A_0 = C[[x_1, \ldots, x_{d-1}]]$, C Cohen (discrete valuation) ring, finite and generically étale (obvious).

Difficulty: we don't want the ramification locus to be, for example, $V(p) \subset X_0$ ($\Rightarrow$no hope to apply Weierstraß, to algebraize).

Solution: use H. Epp's theorem to make X generically étale over the "special fiber" $V(p)$. (This is part of Ofer Gabber's method to prove finiteness of cohomology.)

Algebraization of cohomology class more subtle than the algebraization of the "denominator" f above (uses formal/henselian comparison theorem).

Reminder on Helmut Epp's theorem

Theorem (Helmut Epp, 1973)

Let $T \to S$ a dominant morphism of complete traits, of residue characteristic $p > 0$. Assume the residue field κ_S is perfect and that the maximal perfect subfield of κ_T is algebraic over κ_S. Then there exists a finite extension of traits $S' \to S$ such that

$$T' := (T \times_S S')^{\nu}_{\mathrm{red}}$$

has reduced special fiber over S'.

Helmut Epp
Eliminating wild ramification.
Inventiones mathematicæ, 1973.

Franz-Viktor Kuhlmann
A correction to Epp's paper "Elimination of wild ramification"
Inventiones mathematicæ, 2003.

Mixed-characteristic upper bound: $\mathrm{cd}_p(K) \leq \dim(A) + \dim_p(k)$

Let k_0 the maximal perfect subfield of the residue field k of A (A is complete, normal).

Apply H. Epp's result over $W(k_0)$, at the generic points of $V(p)$, to get reduced special fiber.

Lemma

Let X be a noetherian normal scheme. Every generically reduced Cartier divisor is reduced.

Get by O. Gabber's theorem $X_p \to \mathrm{Spec}(k[[x_1, \ldots, x_{d-1}]])$ finite, generically étale.

Lift to $X \to X_0 := \mathrm{Spec}(C(k)[[x_1, \ldots, x_{d-1}]]$. ($C(k) =$ Cohen ring.)

By construction, ramification locus (downstairs) doesn't contain $V(p)$. Weierstraß$\Rightarrow$ contained in $V(f)$, $f \in C(k)[[x_1, \ldots, x_{d-2}]][x_{d-1}]$ (up to change of coordinates).

Upper bound: $\mathrm{cd}_p(K) \leq \dim(A) + \dim_p(k) = d + r$

Sketch 2/2

We can algebraize $X \to X_0$ (R. Elkik) into $\widetilde{X} \to \widetilde{X_0} = \mathrm{Spec}(C(k)[[x_1, \ldots, x_{d-2}]]\{x_{d-1}\})$.

We want to show that $\mathrm{H}^{d+r+i}(K, \mathbf{Z}/p) = 0$, for all $i > 0$. Choose c.

Extend c to an algebraizable locus of X? (I.e. does it come from $\widetilde{K}$?)

Apply K. Katō's theorem and equal characteristic case in the codimension one point (in the special fiber), to make the "pole" locus small in $V(p)$.

Lemma

Let B be a discrete valuation ring, B^{h} its henselization, and K (resp. K^{h}) the respective fraction fields. If the image of an element $c \in \mathrm{H}^j(\mathrm{Spec}(K)_{\acute{e}t}, \mathbf{Z}/N)$ is zero in $\mathrm{H}^j(\mathrm{Spec}(K^{\mathrm{h}})_{\acute{e}t}, \mathbf{Z}/N)$, then c belongs to the image of the restriction morphism

$$\mathrm{H}^j(\mathrm{Spec}(B)_{\acute{e}t}, \mathbf{Z}/N) \to \mathrm{H}^j(\mathrm{Spec}(K)_{\acute{e}t}, \mathbf{Z}/N).$$

Weierstraß / R. Elkik + Fujiwara-Gabber (+ K. Katō and induction) to conclude.

Reminder on Kazuhiro Fujiwara and Ofer Gabber's theorem

Theorem (Kazuhiro Fujiwara and O. Gabber)

For a noetherian henselian pair $(X = \mathrm{Spec}(A), Y = V(I))$ and a torsion étale sheaf $\mathscr{F}$ on $U := X - Y$, the canonical morphism

$$\mathrm{H}^a(U_{\acute{e}t}, \mathscr{F}) \to \mathrm{H}^a(\widehat{U}_{\acute{e}t}, \widehat{\mathscr{F}})$$

is an isomorphism.

Observe that we don't make hypothesis on the torsion.

Kazuhiro Fujiwara.
Theory of tubular neighborhood in étale topology.
Duke mathematical journal, 1995.

Esta terminado.

Proof of O. Gabber's theorem 1/4

Theorem (O. Gabber, conf. L. Illusie, lemma 8.1)

Let A be an integral local complete noetherian ring of dimension d with residue field k. There exists a subring A_0 of A, isomorphic to $k[[t_1, \dots, t_d]]$ such that $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ is finite, generically étale.

Let κ be a field of representative of k in A (to be changed later) $\longleftrightarrow$ lifting B_i $(i \in I)$ in A of a p-basis b_i of k (hence non unique if k isn't perfect).
For all finite subset $e \subset I$, let $\kappa_e := \kappa^p(B_i,\, i \notin e)$ $\Rightarrow$ filtered decreasing family of cofinite sub-κ^p-extension of κ, such that $\cap_e \kappa_e = \kappa^p$.

Proof of O. Gabber's theorem 2/4

$\kappa_e := \kappa^p(B_i,\, i \notin e)$, B_i lifting p-basis, e finite; $\cap_e \kappa_e = \kappa^p$.

Let $t_1, \ldots, t_d \in A$ a system of parameters
$K_e := \mathrm{Frac}\, \kappa_e[[t_1^p, \ldots, t_d^p]] \subset K_\kappa := \mathrm{Frac}\, \kappa[[t_1, \ldots, t_d]] \subset K$.

$$\bigcap_e K_e = K_\kappa^p \overset{\text{big } e}{\Rightarrow} \mathrm{rk}_K\, \Omega^1_{K/K_e} = \mathrm{rk}_{K_\kappa}\, \Omega^1_{K_\kappa/K_e}.$$

Observe:

- $\mathrm{rk}_K\, \Omega^1_{K/K_e} = \mathrm{rk}_A\, \Omega^1_{A/\kappa_e[[t_1^p,\ldots,t_d^p]]}$ (gen. rank i.e. $\mathrm{rk}_K(K \otimes_A -)$),
- $\mathrm{rk}_{K_\kappa}\, \Omega^1_{K_\kappa/K_e} = d + \mathrm{rk}_\kappa\, \Omega^1_{\kappa/\kappa_e}$

Hence:

$$\mathrm{rk}_A\, \Omega^1_{A/\kappa_e[[t_1^p,\ldots,t_d^p]]} = \dim(A) + \mathrm{rk}_\kappa\, \Omega^1_{\kappa/\kappa_e} = d + \#e \text{ for some finite set } e.$$

(Reminder: $\Omega^1_{\kappa/\kappa_e}$ generated by the dB_i, $i \in e$.)
By changing the lifting B_i of b_i, $i \in e$ (i.e. $B_i \rightsquigarrow B_i + (m_i \in \mathfrak{m}_A)$), we can make the dB_i linearly independent in $\Omega^1_{A/\kappa_e[[t_1^p,\ldots,t_d^p]]} \otimes_A K$.
(Use: $d(\mathfrak{m}_A)$ generically generates Ω^1_A.)

Using the corresponding fields of representatives (still denoted by κ), we achieve:

$$\mathrm{rk}_A \Omega^1_{A/\kappa[[t_1^p,\ldots,t_d^p]]} = \dim(A).$$

Proof of O. Gabber's theorem 3/4

$$\mathrm{rk}_A \Omega^1_{A/R_\kappa} = \dim(A).$$
$$(R_\kappa := \kappa[[t_1, \ldots, t_d]])$$

Let $f_1, \ldots, f_d \in A$ such that the df_i form a basis of $\Omega^1_{A/R_\kappa} \otimes K$. WLOG: $f_i \in \mathfrak{m}_A$.
Take

$$t_i' := t_i^p(1 + f_i).$$

Observe: $dt_i' = t_i^p df_i$.
Then, A is finite, generically étale over the subring $A_0 = \kappa[[t_1', \ldots, t_d']]$.

Proof of O. Gabber's theorem 4/4

This completes the proof of:

Theorem (O. Gabber)

Let A be a local complete noetherian integral ring of dimension d with residue field κ. There exists a subring A_0 of A, isomorphic to $\kappa[[t_1, \ldots, t_d]]$ such that $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ is finite, generically étale.