Azumaya Algebras and Tsen's Theorem: A Talk

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Troughout this talk, k denotes a field and k_s denotes its separable closure. Denote by G_k the absolute Galois group $\mathrm{Gal}(k_s/k)$.

1 First Notions

Definition 1.1. An **Azumaya algebra** over a filed k is a k-algebra A such that $A \otimes_k k_s$ is isomorphic as a k_s-algebra to the matrix algebra $\mathrm{M}_n(k_s)$ for some $n \geq 1$.

In short, an Azumaya algebra is a twist of a matrix algebra.

Proposition 1.2 (Characterizations of Azumaya algebras)**.** *The following conditions on a k-algebra A are equivalent:*

(1) The algebra A is an Azumaya algebra.

(2) There exists a finite separable extension $L \supseteq k$ such that the L-algebra $A \otimes_k L$ is isomorphic to the matrix algebra $\mathrm{M}_n(L)$ for some $n \geq 1$.

(3) The algebra A is a finite-dimensional simple central algebra.

(4) There is a k-algebra isomorphism $A \simeq \mathrm{M}_r(D)$ for some integer $r \geq 1$ and some finite-dimensional central division algebra D over k.

In (4), r and D are uniquely determined by A.

Definition 1.3. The Brauer group of k, denoted by $\mathrm{Br}\, k$, is the group of equivalence classes of Azumaya algebras over k with multiplication induced by tensor products, under the equivalence relation such that $A \sim B$ if $\mathrm{M}_m(A) \simeq \mathrm{M}_n(B)$ as k-algebras for some m, n.

Theorem 1.4 (Skolem-Noether). *For any two k-algebra morphisms f, g from a simple k-algebra A to an Azumaya k-algebra B, there exists $b \in B^\times$ such that $f(x) = bg(x)b^{-1}$ for all $x \in A$.*

Proof. We first cheat the case that B is a matrix algebra, i.e. $B = \mathrm{M}_n(k) = \mathrm{End}_k(k^n)$. Then the morphisms define actions of A on k^n — let V_f and V_g denote k^n with the actions defined by f and g. Since two A-modules with the same dimension are isomorphic, we have an isomorphism $b : V_g \to V_f$ which is an element of $M_n(k)$ satisfying $f(a) \cdot b = b \cdot g(a)$ for all $a \in A$.

In the general case, we consider the morphisms

$$f \otimes 1, g \otimes 1 : A \otimes_k B^{\mathrm{opp}} \to B \otimes_k B^{\mathrm{opp}}.$$

Because $B \otimes_k B^{\mathrm{opp}}$ is a matrix algebra over k, the first part of the proof shows that there exists $b \in B \otimes_k B^{\mathrm{opp}}$ such that

$$(f \otimes 1)(a \otimes b') = b \cdot (g \otimes 1)(a \otimes b') \cdot b^{-1}$$

for all $a \in A, b' \in B^{opp}$. On taking $a = 1$ in this equation, we find that $(1 \otimes b') = b \cdot (1 \otimes b') \cdot b^{-1}$ for all $b' \in B^{\mathrm{opp}}$. Therefore, $b \in C_{B \otimes_k B^{\mathrm{opp}}}(k \otimes B^{\mathrm{opp}}) = B \otimes kk$, i.e. $b = b_0 \otimes 1$ with $b_0 \in B$. On taking $b' = 1$ in this equation, we find that

$$f(a) \otimes 1 = (b_0 \cdot g(a) \cdot b_0^{-1}) \otimes 1$$

for all $a \in A$, and so b_0 is the element sought. □

Corollary 1.5. *All automorphisms of a central simple k-algebra are inner. In particular, the automorphism group of $\mathrm{M}_n(k)$ is $\mathrm{PGL}_n(k) := \mathrm{GL}_n(k)/k^\times I_n$.*

2 Cohomological interpretation of the Brauer group

Proposition 2.1. *For each $r \geq 1$, there is a natural bijection*

$$\frac{\{\textit{Azumaya } k\textit{-algebra of dimension } r^2\}}{k\textit{- isomorphism}} \simeq H^1(G_k, \mathrm{PGL}_r(k_s)).$$

Proof. Let A be an Azumaya k-algebra such that $A \otimes_k k_s \simeq \mathrm{M}_r(k_s)$ via an isomorphism Φ. Define

$$\alpha_\sigma = \Phi^{-1} \circ (\mathrm{id} \otimes \sigma) \circ \Phi \circ (\mathrm{id} \otimes \sigma^{-1}) \in \mathrm{Aut}_{k_s-\mathrm{algebras}}(\mathrm{M}_r(k_s)) \simeq \mathrm{PGL}_r(k_s).$$

We can verify that α is a 1-cocycle. Different choices of Φ actually give cohomologous cocycles. Thus we get an elemnt of $H^1(G_k, \mathrm{PGL}_r(k_s))$ depending only on A. Now we prove that this map is injective. Suppose we have another Azumaya algebra B satisfying $[B:k]=r^2$ and $\Psi: \mathrm{M}_r(k_s) \simeq B\otimes_k k_s$. Since Φ and Ψ give cohomologous cocycles, after changing Φ (i.e. composing it with an automorphism of $\mathrm{M}_r(k_s)$), we can suppose the two cocycles are equal. Then we have $\Psi\Phi^{-1} = (\Psi\Phi^{-1})^\sigma$ for all $\sigma \in G_k$, and thus the k_s-algebra isomorphism $\Psi\Phi^{-1}: A\otimes_k k_s \to B\otimes_k k_s$ restricts to a k-algebra isomorphism $A \simeq B$.

Conversely, given a cocycle $\alpha \in Z^1(G_k, (\mathrm{PGL}_r(k_s)))$, we set

$$A = \{x \in \mathrm{M}_r(k_s) | \alpha_\sigma \circ (\mathrm{id} \otimes \sigma)(x) = x \text{ for all } \sigma \in G_k\}$$

which is the Azumaya k-algebra we want. □

Taking cohomology of the short exact sequence of G_k-modules

$$1 \to k_s^\times \to \mathrm{GL}_r(k_s) \to \mathrm{PGL}_r(k_s) \to 1$$

yields a morphism

$$\Delta_r : H^1(G_k, \mathrm{PGL}_r(k_s)) \to H^2(G_k, k_s^\times).$$

Combined with the previous proposition, we have a group morphism $\delta: \mathrm{Br}\, k \to H^2(G_k, k_s^\times)$.

Proposition 2.2. *The morphism $\delta: \mathrm{Br}\, k \to H^2(G_k, k_s^\times)$ is bijective.*

Proof. The injectivity follows from the cohomological long exact sequence and the fact that $H^1(G_k, \mathrm{GL}_r(k_s)) = 0$. Now we prove the surjectivity. Choose $\alpha \in H^2(G_k, k_s^\times)$, and suppose that the image of α vanish in $H^2(\mathrm{Gal}(k_s, k'), k_s^\times) \simeq H^2(G_s, (k_s \otimes_k k')^\times)$ for some finite extension $k' \subseteq k_s$ of k of degree r. Fix a basis of k' as k-space and we can define a morphism $(k_s \otimes_k k')^\times \to \mathrm{GL}_r(k_s)$ which associates to x the endomorphism of multiplication by x. Then we have the commutative diagram with exact lines:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k_s^\times & \longrightarrow & (k_s\otimes_k k')^\times & \longrightarrow & (k_s\otimes_k k')^\times/k_s^\times & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & k_s^\times & \longrightarrow & \mathrm{GL}_r(k_s) & \longrightarrow & \mathrm{PGL}_r(k_s) & \longrightarrow & 1. \end{array} \tag{2.2.1}$$

Passing to cohomology, we have the following commutative diagram and the result follows:

$$\begin{array}{ccccc} H^1(G_k, (k_s\otimes_k k')^\times/k_s^\times) & \longrightarrow & H^2(G_k, k_s\times) & \longrightarrow & H^2(G_k, (k_s\otimes_k k')^\times) \\ \downarrow & & \| & & \\ H^1(G_k, \mathrm{PGL}_r(k_s)) & \xrightarrow{\Delta_r} & H^2(G_k, k_s^\times) & & \end{array} \tag{2.2.2}$$

□

Now we consider the long exact sequence of cohomology associated to

$$0 \to \mu_n \to k_s^\times \to k_s^\times \to 0.$$

We can then get that $H^2(k, \mu_n) \simeq (\mathrm{Br}\, k)[n]$, where $(\mathrm{Br}\, k)[n]$ denotes the elements in $\mathrm{Br}\, k$ that are n-torsion.

3 Tsen's theorem

Definition 3.1. Let k be a field. Then k is called C_1 if and only if every homogeneous polynomial $f(x_1, \cdots, x_n)$ of degree $d > 0$ in n variables with $n > d$ has a nontrivial zero in k^n.

Proposition 3.2. *If k is C_1, then $\mathrm{Br}\, k = 0$.*

Proof. Let D be a finite-dimensional central division algebra over k, so $[D : k] = r^2$ for some $r \geq 1$. An associated reduced norm form is of degree r in r^2 variables and has no nontrivial zero. This contradicts the C_1 property unless $r = 1$. This holds for all D, so $\mathrm{Br}\, k = 0$. □

Theorem 3.3 (Tsen)**.** *If L is a the function field of a curve over an algebraically closed field k (that is, L is a finitely generated extension of k of transcendence degree 1), then L is C_1.*

Proof. First consider the case where L is purely transcendental, i.e. $L = k(t)$. Let $f \in L[x_1, \cdots, x_n]$ be a homogeneous polynomial of degree $d > 0$, where $n > d$. Multiplying f by a polynomial in $k[t]$ to clear denominators, we may assume that f has coefficients in $k[t]$. Let m be the maximum of the degrees of these coefficients. We use the method of undetermined coefficients. Choose $s \in \mathbb{Z}_{>0}$ large (later we will say how large), introduce new variables y_{ij} with $1 \leq i \leq n$ and $0 \leq j \leq s$, and substitute

$$x_i = \sum_{j=0}^{s} y_{ij} t^j$$

for all i into f, so that

$$f(x_1, \cdots, x_n) = F_0 + F_1 t + \cdots + F_{ds+m} t^{ds+m},$$

where each $F_l \in k[\{y_{ij}\}]$ is a homogeneous polynomial of degree d in $n(s+1)$ variables. Because $n > d$,

$$n(s+1) > ds + m + 1$$

holds for sufficiently large s and k is algebraically closed, the projective dimension theorem implies that the F_l have a nontrivial common zero over k. This means that f has a nontrivial zero over $k[t]$, hence over $k(t)$.

Then it suffices to prove that if L is algebraic over $k(t)$, then L is C_1. Let $f \in L[x_1, \cdots, x_n]$ be a homogeneous polynomial of degree $d > 0$ where $n > d$. Since L is algebraic over $k(t)$, the coefficients of f generate a finite extension of k. Thus we can suppose that L is a finite extension over k. Choose a basis $e_1, \cdots, c_s$ of L over $k(t)$. Introduce new variables y_{ij} and substitute

$$x_i = \sum_{j=0}^{s} y_{ij} e^j$$

for all i into f, so that

$$f(x_1, \cdots, x_n) = F_1 e_1 + \cdots + F_s e_s,$$

where each $F_l \in k[\{y_{ij}\}]$ is a homogeneous polynomial of degree d in sn variables. Now it suffices to find in $k(t)$ a nontrivial zero of the homogeneous polynomial $g(y_{ij}) := N_{L/k}(f)$, which is of degree sd in sn variables. Since $n > d$ and $k(t)$ is C_1, we have the desired result. $\square$