Definitions of Henselian Pairs

Tongmu He[*,†]

March 11, 2019

In this note, a commutative ring A together with an ideal I is called **a pair** (A, I). We denote $\overline{A} = A/I$.

Definition 1: An **étale neighborhood** of a pair (A, I) is a pair (A', I') together with a ring homomorphism $u : A \to A'$ such that

i). u is étale;

ii). $u(I)A' = I'$;

iii). $\overline{u} : A/I \xrightarrow{\sim} A'/I'$.

$$\begin{tikzcd}
\overline{A'} & A' \arrow[l] \\
\overline{A} \arrow[u, "\cong"] \arrow[ur, phantom, "\square"] & A \arrow[l] \arrow[u, "\text{étale}"']
\end{tikzcd}$$

Geometrically, $\operatorname{Spec} A'$ is étale over $\operatorname{Spec} A$ with identity on the closed sub-scheme $\operatorname{Spec} \overline{A}$.

Theorem 1: For a pair (A, I), the followings are equivalent, and at this time we call (A, I) is a **Henselian pair**:

1. (Étale neighborhood) For any étale neighborhood $A \xrightarrow{u} A'$, there is a retraction $A \underset{v}{\overset{u}{\rightleftarrows}} A'$: $v \circ u = \mathrm{id}_A$.

2. (Idempotent) For any integral A-algebra B, $\operatorname{Idemp}(B) \xrightarrow{\sim} \operatorname{Idemp}(B/IB)$.[a]

3. (Hensel) $I \subseteq \operatorname{rad} A$[b]; and for any monic polynomial $F \in A[T]$, if $f = \overline{F} \in \overline{A}[T]$ has factorization $f = gh$ with g, h monic and $(g) + (h) = (1)$, then there is a lift of factorization $F = GH \in A[T]$ such that $\overline{G} = g, \overline{H} = h$ and G, H are monic.

4. (Newton) $I \subseteq \operatorname{rad} A$; and for any monic polynomial $F \in A[T]$, if there is $\overline{a} \in \overline{A}$ s.t. $f(\overline{a}) = 0$ and $f'(\overline{a}) \in \overline{A}^{\times}$, then there is a lift of root $a \in A$ s.t. $F(a) = 0$ and $a \mapsto \overline{a}$.

5. (Gabber) $I \subseteq \operatorname{rad} A$; and for any $F(T) = T^n(T-1) + a_nT^n + \cdots + a_1T + a_0$ with $a_i \in I$ $(0 \leq i \leq n,\ n \geq 1)$, there is $a \in 1 + I$ such that $F(a) = 0$.

*Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
†Email address: hetm15@mails.tsinghua.edu.cn

[a] Idemp(B) denotes the set of idempotents e of B, $e^2 = e$.
[b] rad A denotes the Jacobson radical of A, which is the intersection of all maximal ideas of A.

Facts:

1. If $I \subseteq \text{rad}(A)$, then the only open subset containing $V(I)$ is $\text{Spec}(A)$. (By definition)

2. For affine scheme $\text{Spec}\, A$, its open and closed subset U 1-1 corresponds to idempotent $e \in \text{Idemp}(A)$ by $U = D(e)$. (See Stacks Project, Lemma 10.20.3, tag 00EE)

The steps of the proof:

$$\begin{tikzcd}
(1). \arrow[rr, Rightarrow, "\text{step 2}"'] \arrow[d, Rightarrow, "\text{step 1}"'] & & (2). \arrow[ll, Rightarrow, bend right, "\text{step 6}"'] \\
(3). \arrow[r, Rightarrow, "\text{step 3}"'] & (4). \arrow[r, Rightarrow, "\text{step 4}"'] & (5). \arrow[u, Rightarrow, "\text{step 5}"']
\end{tikzcd}$$

Step 0. "Étale neighborhood" $\Rightarrow$ "Jacobson radical": (1).$\Rightarrow I \subseteq \text{rad}(A)$

In order to apply (1)., we have to consider some examples of étale neighborhoods. The most trivial example is the open immersion which contains $\text{Spec}\, \overline{A}$:

> *Proof.* For any $a \in I$, take $f = 1 + a$, consider the open immersion $A \to A_f$, then it is an étale neighborhood of (A, I) by definition. Then the condition (1). implies there is a retraction $A \leftarrow A_f$, which is equivalent to that $f = 1 + a$ is invertible in A for any $a \in I$. Thus $I \subseteq \text{rad}(A)$. □

Step 1. "Étale neighborhood" $\Rightarrow$ "Hensel": (1).$\Rightarrow$(3).

> *Proof.* In fact, there is a philosophy about the meaning of Étale neighborhood (see Stacks Project, Section 15.9, tag 07LW):
>
> **Many structures over $\overline{A}$ can be lifted**
>
> **over some étale neighborhood of A**
>
> **Lemma 1 (Tag 0ALH):** For any monic polynomial $F \in A[T]$, if $f = \overline{F} \in \overline{A}[T]$ has factorization $f = gh$ with g, h monic and $(g) + (h) = (1)$, then there is an étale neighborhood A' of A and a lift of factorization $F = GH \in A'[T]$ such that $\overline{G} = g, \overline{H} = h$ and G, H are monic.

Using this lemma,

$$\begin{tikzcd}
f = gh & \overline{A'} & A' \arrow[l] \arrow[d, bend left, dotted, "\text{retract by (1).}"] & F = GH \\
f = gh & \overline{A} \arrow[u, "\cong"] & A \arrow[l] \arrow[u] & F
\end{tikzcd}$$

we know (1).⇒(2)..

Proof of Lemma 1. Study these factorization things via a universal viewpoint:

Universal Factorization Algebras (Tag 00UA) Consider polynomial rings

$$R = \mathbb{Z}[\alpha_1, \ldots, \alpha_n] \qquad Q = \mathbb{Z}[\beta_1, \ldots, \beta_d, \gamma_{d+1}, \ldots, \gamma_n] \qquad S = \mathbb{Z}[x_1, \ldots, x_n]$$

The formal factorization

$$T^n + \sum_{i=1}^{n} \alpha_i T^{n-i} = \left(T^d + \sum_{i \le d} \beta_i T^{d-i} \right) \left(T^{n-d} + \sum_{i>d} \gamma_i T^{n-i} \right) = (T - x_1) \cdots (T - x_n)$$

gives ring homomorphisms:

$$\begin{tikzcd}
Q \arrow[r] & S \\
R \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Q[T] \arrow[r] & S[T] \\
R[T] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
GH = \left(T^d + \sum_{i \le d} \beta_i T^{d-i} \right) \left(T^{n-d} + \sum_{i>d} \gamma_i T^{n-i} \right) \arrow[r, mapsto] & (T - x_1) \cdots (T - x_n) \\
F = T^n + \sum_{i=1}^{n} \alpha_k T^{n-i} \arrow[u, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}$$

Remark that the symmetric group $\mathfrak{S}_n$ acts on $\{x_1, \ldots, x_n\}$ by permutation, and then R, Q can be regarded as sub-algebras of S via the map above:

$$\begin{tikzcd}
Q = S^{\mathfrak{S}_{d,n-d}} \arrow[r, hook] & S \\
R = S^{\mathfrak{S}_n} \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

and thus $Q, S, Q/R, S/R$ are free R-modules.

Back to the proof of the lemma, for any polynomial $F = T^n + \sum_{i=1}^n a_i T^{n-i} \in A[T]$, assign the formal coefficients $\alpha_i \mapsto a_i$, we get $R \to A$. Then tensor by A to the universal algebras, we get

$$\begin{tikzcd} A' = A \otimes_R Q & A'[T] \\ A \arrow[u, hook] & A[T] \arrow[u, hook] \end{tikzcd}$$

and by the discussion above, F can be **automatically factorized** into $F = GH$ in $A'[T]$, i.e. $\operatorname{Spec} A'[T]/(F)$ is a disjoint union of two open subsets defined by G and H. Now we are going to **take an open subset of** $\operatorname{Spec} A'$ **which is an étale neighborhood of** A. Let $A' \to \overline{A}$ be the map assigning the formal coefficients β_i, γ_i to the coefficients of g and h, where $\overline{F} = f = gh$. This is obviously surjective, thus

$$\begin{tikzcd} A'/J \arrow[dr, "\cong"'] & A'/IA' \arrow[l] & A' \arrow[l] \\ & A/I \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Since the Jacobian of the map $A \otimes_R R \to A \otimes_R Q$ is the resultant of the two polynomials G, H (see Tag 00UA). Therefore, for any point $\mathfrak{p} \in \operatorname{Spec} A/I = \operatorname{Spec} A'/J \subseteq \operatorname{Spec} A'$, the resultant of G, H at this point is the resultant of $g, h \in \overline{A}[T]$, which is invertible due to $(g) + (h) = (1)$ and the property of resultant. Therefore, A' is étale over[a] A at this point $\mathfrak{p}$.

While the set of non-étale points is a closed subset $V(J')$, thus $J + J' = A'$, we then choose $s \in A'$, such that $V(J) \subseteq \operatorname{Spec} A_s$ and $\operatorname{Spec} A_s$ is étale over A.

$$\begin{tikzcd} A'_s/IA'_s \arrow[d, bend right] & A'_s \arrow[l] \\ A/I \arrow[u, "\text{étale}"'] \arrow[ur, phantom, "\square"] & A \arrow[l] \arrow[u, "\text{étale}"'] \end{tikzcd}$$

Thus the surjective map $A'_s/IA'_s \to A/I$ is also étale. Since étale map is an open map, thus $\operatorname{Spec} A/I$ is an open and closed subset of $\operatorname{Spec} A'_s/IA'_s$, then we get an idempotent e s.t.

$$\begin{tikzcd} A'_{se}/IA'_{se} & A'_{se} \arrow[l] \\ A/I \arrow[u, "\cong"] \arrow[ur, phantom, "\square"] & A \arrow[l] \arrow[u] \end{tikzcd}$$

In conclusion, A'_{se} is an étale neighborhood of A such that the factorization of $\overline{f}$ can be lifted to $F = GH \in A'_{se}[T]$.

[a] Since A' is flat and of finite presentation over A by base change, then Jacobian non-vanishing implies unramification near $\mathfrak{p}$, thus a neighborhood of $\mathfrak{p}$ is étale over A

$\square$

Step 2. "Étale neighborhood" $\Rightarrow$ "Idempotent": (1).$\Rightarrow$(2).

This is a bonus of (1).$\Rightarrow$(3)., which verifies the philosophy mentioned above. More important, this is the critical but technical middle step to prove the equivalence of these definitions.

Proof. Similarly (1).$\Rightarrow$(2). follows this lemma:

Lemma 2 (Tag 07M4): For any integral A-algebra B and any idempotent $\overline{e} \in \mathrm{Idemp}(B/IB)$, there is an étale neighborhood A' of A such that $\overline{e}$ can be lifted to an idempotent $e \in \mathrm{Idemp}(B')$ where $B' = B \otimes_A A'$.

Proof for Lemma 2. Take any lift b of $\overline{e}$ in B, then $b^2 - b \in IB$. Since IB is integral over I, there is a polynomial $F \in A[T]$ such that $F(T) = (T^2 - T)^n + \sum_{i<2n} b_i T^i$ with $b_i \in I$ and $F(b) = 0$. Then $f = \overline{F} = T^n(T-1)^n = gh$ in $\overline{A}[T]$. Thanks to the identities:

$$(T-1) \cdot (T+1)(T^2+1) \cdots (T^{2^n}+1) = T^{2^{n+1}} - 1$$

so that $(g) + (h) = (1)$. Then we may apply Lemma 1 ((1).$\Rightarrow$(3).): there is an étale neighborhood A' of A such that the factorization of f can be lifted to $F = GH \in A'[T]$ with $\overline{G} = g = T^n, \overline{H} = h(T-1)^n$ and G, H monic.
Let $b_1 = G(b) \mapsto \overline{e}^n = \overline{e} \in \overline{A}$, $b_2 = H(b) \mapsto (\overline{e}-1)^n = \pm(1-\overline{e}) \in \overline{A}$, then $b_1 b_2 = F(b) = 0$. **However, b_1, b_2 may not generate the whole B', which means that $D(b_1), D(b_2)$ may not cover $\operatorname{Spec} B'$.** We have to shrink $\operatorname{Spec} B'$ but not to lose $\operatorname{Spec} B'/IB'$.
Anyhow, (b_1, b_2) generates $B'/IB' = B/IB$ by $(\overline{e}, 1-\overline{e})$. Thus $V(b_1, b_2) \cap V(IB') = \varnothing$ in $\operatorname{Spec} B'$. While B' is integral over A', $\operatorname{Spec} B' \to \operatorname{Spec} A'$ is a closed map. Then the images of $V(b_1, b_2)$ and $V(IB')$ in $\operatorname{Spec} A'$ are disjoint closed subset, thus there is $a \in A$ such that $a|_{V(b_1,b_2)} = 0$ while $a|_{V(IB')} = 1$.
Then replace A' by another étale neighborhood A'_a, then (b_1, b_2) generates B'_a and $b_1 b_2 = 0$. This implies $\operatorname{Spec} B'_a = D(b_1) \amalg D(b_2)$. The intersection of this decomposition with the closed subscheme $\operatorname{Spec} B'_a/IB'_a = \operatorname{Spec} B/IB$ induces the decomposition given by the idempotent $\overline{e}$. Thus the idempotent corresponding to $D(b_1)$ lifts $\overline{e}$. $\square$

$\square$

Step 3. "Hensel" $\Rightarrow$ "Newton": (3).$\Rightarrow$(4).

Proof. This is easy.For any monic polynomial $F \in A[T]$, if there is $\overline{a} \in \overline{A}$ s.t. $f(\overline{a}) = 0$ and $f'(\overline{a}) \in \overline{A}^{\times}$, then $f(T) = (T - \overline{a})h(T) \in A[T]$ where $(T - \overline{a}) + (h) = (1)$. Apply (2)., then we have a lift $F = GH$ where G is monic and $\overline{G} = T - \overline{a}$. Hence there is $a \in A$ s.t. $F(a) = 0$ and $a \mapsto \overline{a}$. $\square$

Step 4. "Newton" $\Rightarrow$ "Gabber": (4).$\Rightarrow$(5).

Proof. This is direct. $\square$

Step 5. "Gabber" $\Rightarrow$ "Idempotent": (5).$\Rightarrow$(2).

Proof. For any integral A-algebra B and any idempotent $\overline{e} \in \mathrm{Idemp}(B/IB)$, take any lift b of $\overline{e}$ in B. Without loss of generality, we may assume B is generated by b as A-algebra, then B/IB is generated by the idempotent $\overline{e}$, i.e. $B/IB = A\overline{e} \times A1 \cong C_1 \times C_2$ where $\overline{e}$ corresponds to $(1,0)$ in $C_1 \times C_2$.

By Lemma 2 ((1).⇒(2).), there is an étale neighborhood A' of A such that $\overline{e}$ can be lifted to an idempotent $e' \in \mathrm{Idemp}(B')$ where $B' = B \otimes_A A'$. Thus $B' \cong B'_1 \times B'_2$, where we may assume $e' = (1, 0)$.
Consider the image of b in B': $b' = b \otimes 1 = (b'_1, b'_2) \in B'_1 \times B'_2$. By the definition of liftness, we have $b'_1 - 1 \in IB'$, $b'_2 \in IB'$.
On the one hand, $A'/IA' \to B'_1/IB'_1 = C_1$ is surjective (by the definition of C_1), thus $I \subseteq \mathrm{rad}(A)$ implies by Nakayama lemma that $A' \to B'_1$ is surjective. Thus we may take $a' \mapsto b'_1 - 1$.
On the other hand, IB' is integral over I, thus there is a polynomial $w(T) = T^n + \sum_{i<n} c_i T^i$ with $c_i \in I$ such that $w(b'_2) = 0$.
Then $b' = (b'_1, b'_2)$ is a root of $(T - 1 - a') \cdot w(T)$ which indicates that

$$(b' - 1)b'^n \in \sum_{i \leq n} IA' \cdot b'^i$$

Since $A \to A'$ is étale, thus flat and open; and $I \subseteq \mathrm{rad}(A)$, thus $\operatorname{Spec} A' \to \operatorname{Spec} A$ is surjective, hence $A \to A'$ is faithfully flat. We thus descend the relation for b':

$$(b - 1)b^n \in \sum_{i \leq n} I \cdot b^i$$

that is, there is a polynomial $F(T) = (T - 1)T^n + \sum_{i \leq n} a_i T^i$ with $a_i \in I$ such that $F(b) = 0$. Then we may apply Gabber condition (5.) on this polynomial, we get an $a \in 1 + I$ such that $F(a) = 0$. Then

$$\begin{tikzcd} T \arrow[d, mapsto] & A[T]/(F) \arrow[r, "\cong"] \arrow[d, two heads] & A[T]/(T-a) \times C \arrow[dd] & (T,0) \arrow[dd, mapsto] \\ b \arrow[d, mapsto] & B \arrow[d, two heads] & & \\ \overline{e} & B/IB \arrow[r, "\cong"] & C_1 \times C_2 & (1,0) \end{tikzcd}$$

This forces the closed sub-scheme $\operatorname{Spec} B$ in $\operatorname{Spec} A[T]/(F)$ admitting a decomposition of two disjoint open subsets which is compatible with the decomposition of $\operatorname{Spec} B/IB$. This completes the proof. □

Step 6. "Idempotent" ⇒ "Étale neighborhood": (2).⇒(1).

Proof. For any étale neighborhood $A \xrightarrow{u} A'$, in order to apply (2)., we have to take the integral closure B of A in A':

$$\begin{tikzcd} \overline{A'} & & A' \arrow[ll] \\ B/IB \arrow[u] & & B \arrow[u] \\ \overline{A} \arrow[u] \arrow[uu, bend left, "\cong"'] & & A \arrow[ll] \arrow[u] \end{tikzcd}$$

Then A' is quasi-finite at the point $\mathfrak{p} \in \operatorname{Spec} \overline{A'}$. By Zariski main theorem, the morphism $\operatorname{Spec} A' \to \operatorname{Spec} B$ is an open immersion near $\mathfrak{p}$. Let $U = \operatorname{Spec} A'_s$ be the union of such neighborhoods of all $\mathfrak{p} \in \operatorname{Spec} \overline{A'}$. Then $\operatorname{Spec} \overline{A'} = \operatorname{Spec} \overline{A}$ is an open subset in $\operatorname{Spec} B/IB$,

$$\begin{tikzcd} U \arrow[r, "\circ"] & \operatorname{Spec} B \\ \operatorname{Spec} \overline{A'} \arrow[u, "\prime"] \arrow[r, "\circ"] & \operatorname{Spec} B/IB \arrow[u, "\prime"] \end{tikzcd}$$

hence $\operatorname{Spec} B/IB = \operatorname{Spec} \overline{A'} \amalg D$, this gives an idempotent of B/IB. Apply the Idempotent condition (2)., we get a decomposition $B = B_1 \times B_2$. Hence $A'_s = A'_1 \times A'_2$. Then A'_1 is integral over A (open subset of B) and also étale over A, thus A'_1 is finite over A and also finite presented and flat over A. Then A'_1 is locally free of finite rank over A, while $A'_1/IA'_1 = A/I$ by the discussion above. Hence $A'_1 \cong A$, which gives us the retraction $A' \to A'_s \to A'_1$. □

Appendix: "Newton" ⇒ "Hensel": (4).⇒(3).

There is an elementary proof for this direction, and one may see some clues of Gabber's result:

Crépeaux's proof. For any monic polynomial $F \in A[T]$, if $f = \overline{F} \in \overline{A}[T]$ has factorization $f = gh$ with g, h monic and $(g) + (h) = (1)$, consider the universal factorization algebras $Q_A = A \otimes_R Q, S_A = A \otimes_R S$:

$$\begin{tikzcd} Q_A = S_A^{\mathfrak{S}_{d,n-d}} \arrow[r] & S_A \\ A = S_A^{\mathfrak{S}_n} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Consider $Z = \prod_{\substack{i \le d \\ j > d}} (x_i - x_j) \in S_A^{\mathfrak{S}_{d,n-d}} = Q_A$, then

$$P(Y) = \prod_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{d,n-d}} (Y - Z^\sigma) \in S_A^{\mathfrak{S}_n}[Y] = A[Y]$$

Denote the formal lift of factorization $f = gh$ by $F = GH \in Q_A[T]$, **the strategy is using $P(Y)$ as a modifier to descend this factorization down to $A[T]$:**

$$\left(\sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{d,n-d}} \frac{P(Y)}{Y - Z^\sigma} G^\sigma \right) \cdot \left(\sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{d,n-d}} \frac{P(Y)}{Y - Z^\sigma} H^\sigma \right) \tag{1}$$

$$= P'(Y)^2 F - P(Y) \left(\sum_{\sigma \neq \tau} \frac{P(Y)}{(Y - Z^\sigma)(Y - Z^\tau)} (F - G^\sigma H^\tau) \right) \tag{2}$$

Since the image of Z in $Q_{\overline{A}}$ is $z = h(x_1) \cdots h(x_d)$, thus the image of P in $\overline{A}[Y]$ is $Y^{\binom{n}{d}-1}(Y - z)$. Therefore, "Newton" (3). implies there is $a \in A$ such that $P(a) = 0$, $P'(a) \in A^\times$ and $a \mapsto z$. Let G_Y, H_Y denote the two terms in the parentheses in the first line in the equation (1), and we get:

$$F = \left(P'(a)^{-1} G_a\right) \cdot \left(P'(a)^{-1} H_a\right) \in A[T]$$

where the images of the product in $\overline{A}[T]$ are exactly $g \cdot h$. This completes the proof. □

Discussion

Question 1: Now that we have a short proof for (4).⇒(3)., **why couldn't we directly prove "Hensel" ⇒ "Idempotent"** ((3).⇒(2).) so that we don't need to care about (5).?

Answer 1: If we start from "Hensel", we get the lift of decomposition $F = GH$. However, it may not be true that $(G) + (H) = (1)$, which indicates that G, H may not give a disjoint union of the space. We have already discussed something in Step 2. "Étale neighborhood" ⇒ "Idempotent": (1).⇒(2)., in which place we have to **shrink the space** so that G, H could give the decomposition of the space. However, as for the lift of idempotents to B, the space $\operatorname{Spec} B$ is already given, which can't be changed naively, hence we might not get (3).⇒(2). directly.

Question 2: But you also "shrink the space" in Step 6. "Idempotent" ⇒ "Étale neighborhood": (2).⇒(1)., why we succeeded here even if we shrank the space while we couldn't if we shrink the space for "Idempotent" (2).?

Answer 2: Well, I think it is the most subtle thing of these equivalence conditions, it seems like (1). is more flexible than (2). since it allows us to shrink the space (well I'm just stating the phenomena). If we open up the regression process of prove (3).⇒(2). via (3).⇒(1).⇒(2)., you will find that in order to lift the idempotent:

1. we first lift it to an étale neighborhood (Lemma 2) (lift once and shrink once);
2. for this étale neighborhood and the idempotent, we shrink it again in order to make $F = GH$ corresponds to decomposition of space ((3).⇒(1)., which is not written in this note);
3. the decomposition gives a retraction, so the it descends.

Hence you see, we didn't apply "Hensel" for the "Idempotent" directly, but apply it to a bigger space, namely the étale neighborhood.

Question 3: We have seen some potential relations between Crépeaux's proof and Gabber's, how do you explain their relationship?

Answer 3: Actually, the common of the two approaches is the use of universal factorization algebras, which was locked into a black box (Lemma 1) in Gabber's approach. Hence we can simply think that the concerned étale neighborhood and A-algebra B are just:

$$\begin{array}{ccc} A' = A \otimes_R Q & \qquad & B' = B \otimes_A A' \\ \uparrow & & \uparrow \\ A & & B \end{array}$$

Recall that the decompositions are for free on the top line.

Briefly, Gabber's approach (Step 5. "Gabber" $\Rightarrow$ "Idempotent": (5).$\Rightarrow$(2).) is: for any lift b of the idempotent $\bar{e}$, it satisfies a polynomial of the form $T^n(T-1)+I[T]$. This is proved by the faithfully flatness of $A \to A'$. Geometrically, it can be regarded as the **"A-surjectivity" of** $\operatorname{Spec} B' \to \operatorname{Spec} B$ **sends the automatic decomposition down**.

While in Crépeaux's proof, he constructed the retraction modified by $P(Y)$ from the universal factorization algebra B' to B. It seems that geometrically $\operatorname{Spec} B$ **is being put into** $\operatorname{Spec} B'$ **via section modified by** $P(Y)$ so that it is compatible with the given decomposition of B/IB. However, this plausible algebraic view has big trouble, since the retraction map $G \mapsto \sum \frac{P(Y)}{Y-Z^\sigma} G^\sigma$ is far away from a ring homomorphism. But it went back to algebraic if we get a simple root for $P(Y)$, when the identity with remainders collapsed to $F = GH$. It seems like what an analyst would do.

In conclusion, they are totally two different directions. (But there is some sense of duality of them, isn't it?)

References

[1] E. Crépeaux, *Une caractérisation des couples henséliens*, Enseignement Math. **13** (1967), 273–279 (1968). MR 0237484

[2] Fabrice Orgogozo, *Corps finis II*, `http://fabrice.orgogozo.perso.math.cnrs.fr/articles/Shanghai1.pdf`, 2015.

[3] Michel Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Mathematics, Vol. 169, Springer-Verlag, Berlin-New York, 1970. MR 0277519

[4] The Stacks project authors, *The stacks project*, `https://stacks.math.columbia.edu`, 2019.