Artin's comparison theorem via $K(\pi, 1)$

Zhixiang Wu

May 5, 2019

We aim to prove Lemma 4.5 of XI, [3]:

Lemma 0.1. *If $\xi \in H^q(X_{cl}, F)$ and $x \in X_{cl} = X(\mathbb{C})$. Then there exists an étale morphism $X' \to X$ whose image containing x such that the image of ξ in $H^q(X'_{cl}, F)$ is zero.*

Firstly, the problem is Zarisiki local since open immersions are étale. We can assume X is an Artin neighborhood of x since such a (Zariski) neighborhood exists by Proposition 3.3 of XI, [3], that is there is a sequence of fibrations:

$$X = X_n, \cdots, X_0 = \text{Spec}(\mathbb{C})$$

where for each $0 \le i \le n-1$, there is a morphism $\pi_i : X_{i+1} \to X_i$ which is an elementary fibration. Let $\pi = \pi_1(X(\mathbb{C}), x)$ be the fundamental group of $\pi_1(X(\mathbb{C}), x)$. We will prove that $X(\mathbb{C})$ is a $K(\pi, 1)$ space. We now give the definition of $K(\pi, 1)$.

1 $K(\pi, 1)$

Let Y be a (good) connected topology space (for example, a connected manifold) and F be a local system (locally constant sheaf of abelian groups) on Y. Let $y \in Y$ be a point. Then the monodromy actions equip fiber F_y with a structure of a (right) module of $\pi_1(Y, y)$. This gives a functor from the category of local systems on Y to the category of $\pi_1(Y, y)$-modules. The functor induces an equivalence of the two categories. If M is a $\pi_1(Y, y)$-module, we let F_M be the corresponding local system with fiber $F_{M,y} \simeq M$ as $\pi_1(Y, y)$-modules. Since $M^{\pi_1(Y,y)} = \Gamma(Y, F_M)$, we can consider derived functors of $\Gamma(Y, -)$ in the category of local systems (which is equivalent to $\pi_1(Y, y)$-modules) and category of sheaves on Y. By the formalism of universal δ-functors, we have natural morphisms of cohomology groups:

$$\rho^i : H^i(\pi_1(Y, y), M) \to H^i(Y, F_M)$$

for any $i \ge 0$. These morphisms are not necessarily isomorphisms. Let π be a group.

Definition 1.1. *A connected topological space Y is called Eilenberg-Maclane of type $K(\pi, 1)$ if $\pi_1(Y) \simeq \pi$ and homotopy group $\pi_i(Y)$ is trivial for any $i \ge 2$.*

Proposition 1.2. *Y is $K(\pi_1(Y,y),1)$ if and only if the morphism ρ^i above is an isomorphism for any i and any $\pi_1(Y,y)$-module M.*

Proof Let $p : \widetilde{Y} \to Y$ be the universal cover of Y. Assume that Y is $K(\pi_1(Y,y),1)$. Then $\pi_i(\widetilde{Y}) \simeq \pi_i(\widetilde{X}) = 0$ when $i \geq 2$. By Hurewicz theorem, which asserts that the first non-trivial homotopy group and homology group are isomorphic (if π_1 is trivial), we get $H_i(\widetilde{Y},\mathbb{Z}) = 0$ for any $i \geq 1$. If F is an arbitrary local system over Y, p^*F is constant since $\widetilde{Y}$ is simply connected. Then $H^i(\widetilde{Y}, p^*F) = 0$ for any $i \geq 1$ by the universal coefficient theorem. We have a (Hoschild-Serre) spectral sequence:

$$E_2^{p,q} = H^p(\pi_1(Y,y), H^q(\widetilde{Y}, p^*F)) \Rightarrow H^{p+q}(Y,F).$$

Hence the spectral sequence degenerates and ρ^i are isomorphisms.

(The spectral sequence is a Grothendieck spectral sequence associated to functors $\Gamma(\widetilde{Y}, p^*(-))$ from the category of sheaves over Y to the category of $\pi_1(Y,y)$-modules and $(-)^{\pi_1(Y,y)}$ from the category of $\pi_1(Y,y)$-modules to the category of abelian groups. $\Gamma(\widetilde{Y}, p^*(-))$ maps an injective sheaf I to an acyclic $\pi_1(Y,y)$-module since the functor $M \mapsto \mathrm{Hom}_{\pi_1(Y,y)}(M, \Gamma(\widetilde{Y}, p^*I)) = \mathrm{Hom}(\underline{M}_{\widetilde{Y}}, p^*I)^{\pi_1(Y,y)} = \mathrm{Hom}(F_M, I)$ is exact.)

Conversely, assume ρ^i are isomorphisms. Let $\underline{\mathbb{Z}}_{\widetilde{Y}}$ be the constant sheaf over $\widetilde{Y}$ with coefficient in $\mathbb{Z}$. Then $p_*\underline{\mathbb{Z}}_{\widetilde{Y}}$ is a local system over Y. We have $H^i(\widetilde{Y}, \underline{\mathbb{Z}}_{\widetilde{Y}}) \simeq H^i(Y, p_*\underline{\mathbb{Z}}_{\widetilde{Y}}) \simeq H^i(\pi_1(Y,y), (p_*\underline{\mathbb{Z}}_{\widetilde{Y}})_y) = 0$ if $i \geq 1$. By Hurewicz theorem again, $\widetilde{Y}$ is $K(\pi,1)$. Then the same holds for Y since Y and $\widetilde{Y}$ have same higher homotopy groups. □

Proposition 1.3. *If $x \in X$ such that X is an Artin neighborhood of x over Spec$(\mathbb{C})$, then $X(\mathbb{C})$ is $K(\pi_1(X(\mathbb{C}),x),1)$ and $\pi_1(X(\mathbb{C}),x)$ is a succession of extensions of free groups of finite type.*

Proof By hypotheses, there is a sequence of fibrations:

$$X = X_n, \cdots, X_0 = \mathrm{Spec}(\mathbb{C})$$

and for each $0 \leq i \leq n-1$, $\pi_i : X_{i+1} \to X_i$ is an elementary fibration. By induction, we only need to show if $\pi : X \to S$ is an elementary fibration and S is $K(\pi,1)$ with required fundamental group, then so is X. By the definition of elementary fibrations, there is an embedding $j : X \hookrightarrow \overline{X}$ of S-schemes, with complement $Y \hookrightarrow \overline{X}$ of X in $\overline{X}$ finite étale over S, such that $\overline{X}$ is projective smooth over S with fibers geometrically connected irreducible of dimension 1. Then $X(\mathbb{C})$ is a locally trivial (topological) fiber bundle over $S(\mathbb{C})$ with fiber F, where F is topologically a connected compact surface with a non-empty set of points removed (we can use Ehresmann's theorem to show that $\overline{X}(C)$ is a locally trivial fiber bundle over $S(\mathbb{C})$ firstly). Since fiber bundles are Serre fibrations, we have a long exact sequence of fibration $F \to X(\mathbb{C}) \to S(\mathbb{C})$:

$$\cdots \to \pi_i(F) \to \pi_i(X(\mathbb{C})) \to \pi_i(S(\mathbb{C})) \to \cdots.$$

To conclude, we just need to recall elementary facts that F is a $K(\pi,1)$ and its fundamental group is free of finite type. □

2 Good groups

If G is an abstract group, its profinite completion $\widehat{G}$: $\varprojlim_{H \lhd G, [G:H]<\infty} G/H$, where H runs through all finite index normal subgroups of G, is a profinite group. If M is a (discrete) finite $\widehat{G}$-module, the cohomology of M in the category of continuous $\widehat{G}$-modules is given by $H^i(\widehat{G}, M) = \varinjlim_{H \lhd G, [G:H]<\infty} H^i(G/H, M^H)$. Any such M is naturally a G-module. Thus we have natural morphisms of cohomology groups (by formalism of δ-functors)

$$H^i(\widehat{G}, M) \to H^i(G, M)$$

for $i \geq 0$.

Definition 2.1. *A group G is called good if for any finite G-module M, the morphism $H^i(\widehat{G}, M) \to H^i(G, M)$ is an isomophism for any $i \geq 0$.*

Proposition 2.2 (Charpter 1, 2.6, Exercise 2, [4])**.** *If G is a succession of extensions of free groups of finite type, then G is good.*

We firstly show that Proposition 2.2 implies Lemma 0.1

Proof of Lemma 0.1 Assume $\xi \in H^q(X_{cl}, F)$ and $x \in X_{cl} = X(\mathbb{C})$. By Proposition 1.3 and Proposition 2.2, $X(\mathbb{C})$ is $K(\pi, 1)$ and $\pi := \pi_1(X(\mathbb{C}), x)$ is good. Thus, we have isomophisms of cohomology groups

$$H^q(X(\mathbb{C}), F) \simeq H^q(\pi, F_x) \simeq H^q(\widehat{\pi}, F_x) = \varinjlim_{H \lhd \pi, [\pi:H]<\infty} H^i(\pi/H, F_x^H).$$

Assume the image of ξ in the last term lies in $H^i(\pi/H, F_x^H)$ for some H. By the equivalence of finite étale sites over X and $X(\mathbb{C})$, or equivalently, $\pi_1(X, x)$ is the profinite completion of $\pi_1(X(\mathbb{C}), x)$, we can assume H corresponds to a finite étale cover X' of X (then a finite étale cover $X'(\mathbb{C})$ of $X(\mathbb{C})$) by the Galois correspondence. Let $p : X' \to X$ be the covering map. The image of ξ under the map $H^i(\pi/H, F_x^H) \to H^i(H, F_x)$ is zero when $i \geq 1$, hence the image of ξ in $H^i(X'(\mathbb{C}), p^*F)$ under maps

$$H^i(\pi/H, F_x^H) \to H^i(H, F_x) \to H^i(X'(\mathbb{C}), p^*F)$$

is zero. □

We now do the exercise Proposition 2.2 in Serre's book. Let G be a group.

Lemma 2.3. *The following conditions are equivalent:*

(1) $H^q(\widehat{G}, M) \to H^q(G, M)$ is a bijection for any $q \geq 0$ and any finite discrete $\widehat{G}$-module M.

(2) for any $q \geq 1$, any finite discrete $\widehat{G}$-module M and any $x \in H^q(G, M)$, there exists a finite index subgroup G_0 of G, such that the image of x in $H^q(G_0, M)$ is zero.

Proof (1)⇒(2): we have $H^q(\widehat{G}, M) = \varinjlim_{H \lhd G, [G:H]<\infty} H^q(G/H, M^H)$ and the image of $H^q(G/H, M^H)$ in $H^q(H, M)$ is zero for any H and $q \geq 1$.

(2)⇒(1): induction on q. $q = 0$ is automatic. Assume $H^q(\widehat{G}, M) \to H^q(G, M)$ is an isomorphism for $q \leq n-1$. Let x be an arbitrary element of $H^q(G, M)$. Assume the image of x in $H^q(G_0, M)$ is zero. We have unit morphism $M \to \mathrm{Ind}_{G_0}^G M$, where $\mathrm{Ind}_{G_0}^G M$ is the induced representation. Then the image of x in $H^q(G, \mathrm{Ind}_{G_0}^G M) = H^q(G_0, M)$ is zero. We have long exact sequences associated with short exact sequence $0 \to M \to \mathrm{Ind}_{G_0}^G M \to (\mathrm{Ind}_{G_0}^G M)/M \to 0$:

$$\begin{array}{ccccccc} H^{q-1}(\widehat{G}, (\mathrm{Ind}_{G_0}^G M)/M) & \longrightarrow & H^q(\widehat{G}, M) & \longrightarrow & H^q(\widehat{G}, \mathrm{Ind}_{G_0}^G M) & \longrightarrow & H^q(\widehat{G}, (\mathrm{Ind}_{G_0}^G M)/M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{q-1}(G, (\mathrm{Ind}_{G_0}^G M)/M) & \longrightarrow & H^q(G, M) & \longrightarrow & H^q(G, \mathrm{Ind}_{G_0}^G M) & \longrightarrow & H^q(G, (\mathrm{Ind}_{G_0}^G M)/M) \end{array}$$

Since G_0 has finite index in G, $(\mathrm{Ind}_{G_0}^G M)/M$ is a finite G-module. By the induction hypothesis, $H^{q-1}(\widehat{G}, (\mathrm{Ind}_{G_0}^G M)/M) \to H^{q-1}(G, (\mathrm{Ind}_{G_0}^G M)/M)$ is an isomophism. Chasing diagram, we see x lies in the image of map $H^q(\widehat{G}, M) \to H^q(G, M)$. Since x is arbitrary, $H^q(\widehat{G}, M) \to H^q(G, M)$ is a surjection. To prove the injectivity, let $x \in H^q(G/G_0, M^{G_0})$ such that its image in $H^q(G, M)$ is zero. We consider the same diagram as above. The image of x in $H^q(\widehat{G}, M)$ lies in the image of $H^{q-1}(\widehat{G}, (\mathrm{Ind}_{G_0}^G M)/M)$. A diagram chasing and the induction hypotheses for $q-1$ on modules $\mathrm{Ind}_{G_0}^G M$ and $(\mathrm{Ind}_{G_0}^G M)/M$ show that the image of x in $H^q(\widehat{G}, M)$ is zero. □

Lemma 2.4. *Let G be a discrete group and G_0 be a subgroup of G of finite index. Then G is good if and only if G_0 is good.*

Proof Assume G_0 is good. Let M be a arbitrary finite G-module. If $\xi \in H^q(G, M), q > 0$. By Lemma 2.3, the image of ξ in $H^q(G_0, M)$ will vanish if restrict to a finite index subgroup of G_0, which is also a finite index subgroup of G. By Lemma 2.3, G is good.

Assume G is good. For any finite G_0-module M, $\mathrm{Ind}_{G_0}^G M$ is also finite. Then $H^q(G_0, M) = H^q(G, \mathrm{Ind}_{G_0}^G M) \simeq H^q(\widehat{G}, \mathrm{Ind}_{G_0}^G M) = H^q(\widehat{G_0}, M)$. Thus G_0 is also good. □

Lemma 2.5. *If G is a good group and E is an extension of G by a group N such that N is finite, then there exists a subgroup E_0 of E of finite index such that $E_0 \cap N = \{e\}$, where e is the identity element.*

Proof Let I be the centralizer of N in E. Consider the morphism of groups $E \to \mathrm{Aut}(N)$ given by conjugation of elements of E on N. We see the kernel is I. Since N is finite, I has finite index in E. Then $I/(I \cap N)$ is isomorphic to a finite index subgroup of G. By Lemma 2.4, $I/(I \cap N)$ is good. The exact sequence

$$0 \to I \cap N \to I \to I/(I \cap N) \to 0$$

is a central extension of $I/(I \cap N)$ by I. Classes of central extensions are classified by the cohomology group

$$H^2(I/(I \cap N), I \cap N).$$

Since $I/(I \cap N)$ is good, by Lemma 2.3, the extension above splits when restricted to a finite index subgroup of $I/(I \cap N)$, which, by the splitness, gives out a finite index subgroup E_0 of I such that $E_0 \cap N = \{e\}$. Moreover, E_0 is also a finite index subgroup of E. □

Lemma 2.6. *If N is a finitely generated group and E is an extension of a group G by N. Assume G is good, then any finite index subgroup of N contains a finite subgroup of the form $N \cap E_0$, where E_0 is a finite index subgroup of E. Hence we have an exact sequence of*

$$0 \to \widehat{N} \to \widehat{E} \to \widehat{G} \to 0.$$

Proof Assume N_0 is a finite index subgroup of N. Replace N by $\cap_{g \in N/N_0} gNg^{-1}$, we can assume N_0 is normal in N. Since N is finitely generated, the number of (normal) subgroups of N with fixed index is finite. (Assume index n is fixed. A normal subgroup N' of index n gives out a morphism $N \to S_n$ by the action of N on the cosets N/N',where S_n is the n-th symmetric group. N' is the kernel of the morphism. Since N is finitely generated, the number of morphism from N to S_n is finite.) Thus the set $\{gN_0g^{-1} | g \in E\}$ is finite and $\cap_{g \in E} gN_0g^{-1}$ is a normal subgroup of E which has finite index in N. Hence we can assume N_0 is normal in E. Apply Lemma 2.5 to extension

$$0 \to N/N_0 \to E/N_0 \to G \to 0,$$

we can find a finite index subgroup E'_0 of E/N_0 such that $E'_0 \cap (N/N_0) = \{e\}$. Now let E_0 be the pre-image of E'_0 in E. Then $E_0 \cap N = N_0$. □

Lemma 2.7. *If N is a finitely generated group and E is an extension of a group G by N. Assume G, N is good and $H^q(N, M)$ is finite for any finite N-module M and $q \geq 0$. Then E is also good.*

Proof Since we have exact sequences

$$0 \to N \to E \to G \to 0$$
$$0 \to \widehat{N} \to \widehat{E} \to \widehat{G} \to 0,$$

we have associated (Hoschild-Serre) spectral sequences, for any finite E-module M,

$$\begin{array}{ccc} E_2^{pq} = H^p(\widehat{G}, H^q(\widehat{N}, M)) & \Rightarrow & H^{p+q}(\widehat{E}, M) \\ \downarrow & & \downarrow \\ E_2^{pq} = H^p(G, H^q(N, M)) & \Rightarrow & H^{p+q}(E, M) \end{array}$$

By assumptions, the E_2 terms of the two spectral sequences are isomorphic. Hence the two spectral sequences are isomorphic and $H^q(\widehat{E}, M) \simeq H^q(E, M)$ for any M, q. Thus E is good. □

Proof of Proposition 2.2 By the lemma above, we only need to show that a free group of finite type is good and its cohomology groups of finite modules are finite. Let G be a free group of finite type. We know that a finite group has cohomological dimension 1. (We have a free resolution of trivial G-module $\mathbb{Z}$:

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{g \to 1} \mathbb{Z} \to 0,$$

where I_G is the free $\mathbb{Z}[G]$-module generated by $g_i - 1, i = 1, \cdots, s$, where $(g_1, \cdots, g_s)$ is a choice of free generators of G) Let M be a finite G-module. If $q = 0$, $H^0(\widehat{G}, M) = H^0(G, M) = M^G$ is finite. If $q = 1$, the morphism $H^1(\widehat{G}, M) \to H^1(G, M)$ is in fact an isomophism for any G (even G is not free) and finite module M. If G_0 is a normal finite index subgroup of G, we have a well-known exact sequence (from the Hoschild-Serre spectral sequence)

$$0 \to H^1(G/G_0, M) \to H^1(G, M) \to \cdots.$$

We get an injectivity $H^1(\widehat{G}, M) \hookrightarrow H^1(G, M)$. Now we prove surjectivity. $H^1(G, M)$ are just classes of cross homomorphisms:$f : G \to M, f(gh) = gf(h) + f(g)$. If $f \in Z^1(G, M)$, let $K = \{g | f(g) = 0\}$. Then K is a subgroup of G and it has finite index ($f(gh) = f(g)$ if and only if $h \in K$, so $f : G/K \hookrightarrow M$). Let K_0 be a finite index subgroup of K which is normal in G and take $L = K_0 \cap G_0$, where G_0 is the normal subgroup of elements of G which act trivially on M. Then f comes from an element of $Z^1(G/L, M^L)$. Hence f lies in the image of $H^1(\widehat{G}, M)$. Since f is arbitrary, $H^1(\widehat{G}, M) \to H^1(G, M)$ is surjective. Now G is finitely generated and free, a cross morphism is determined by its value on generators of G, so there are only a finite number of cross morphisms. Hence H^1 is finite. If $q \geq 2$, $H^q(G, M) = 0$ for any finite module M. The condition (2) in Lemma 2.3 is satisfied for G. Hence G is good and $H^q(\widehat{G}, M) = H^q(G, M) = 0$ if $q \geq 2$. □

Remark 2.8. *The Stalling-Swan theorem asserts that a finitely generated group has cohomological dimension* 1 *if and only if it is free.*

Remark 2.9. *The comparisons we have studied are comparisons of cohomology with respect to different topoi. For example, in algebraic setting, let X be a connected, quasi-compact and quasi-separated scheme and let $X_{\text{fét}}$ be the topos corresponding to finite étale covers of X. Let x be a geometric point of X and $\pi_1(X, x)$ be the étale fundamental group. The classifying topos $B\pi_1(X, x)$ is the category of continuous $\pi_1(X, x)$-sets. Then we have an equivalence of topoi $X_{\text{fét}} \simeq B\pi_1(X, x)$ and for any sheaf of abelian group F over $X_{\text{fét}}$, $H^q(B\pi_1(X, x), F_x) \simeq H^q(X_{\text{fét}}, F)$ for any q. See [2] and Chapter 2 of [1]*

References

[1] Piotr Achinger. K(pi, 1) spaces in algebraic geometry. `http://achinger.impan.pl/thesis.pdf` (visited on 2019-04-22).

[2] Piotr Achinger. A theorem of m. artin. MathOverflow. `http://mathoverflow.net/questions/275876/a-theorem-of-m-artin` (visited on 2019-04-22).

[3] N Bourbaki, M Artin, A Grothendieck, P Deligne, and JL Verdier. *Theorie des Topos et Cohomologie Etale des Schemas. Seminaire de Geometrie Algebrique du Bois-Marie 1963-1964 (SGA 4)*, volume 3. Springer, 2006.

[4] Jean-Pierre Serre. *Galois cohomology*. Springer Science & Business Media, 2013.