Absolute flatness & Chevalley's theorem on constructible sets

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These are the notes of the talk I gave on 25 March. The aim was to prove Chevalley's theorem on constructible sets in the affine case using the machinery of absolute flatness. Most of section 1 introduces standard material regarding absolutely flat rings, and can be found (mostly as exercises) in chapters I and II of [Bou06]. Section 2 gives a brief overview of the constructible topology on a scheme, summarising the results explained in much greater generality in §9 of [Gro61]. Section 3 is essentially the proof of Chevalley's theorem, following quite closely the one given in [Oli78].

1 Absolute flatness

1.1 Absolutely flat rings

Definition 1.1. A ring A is **absolutely flat** if every A-module is flat.

Proposition 1.1. Let A be a ring. The following are equivalent:

1. A is absolutely flat.
2. For all $a \in A$, $(a^2) = (a)$.
3. For all $a \in A$ there is a unique $x \in A$ such that $xax = x$ and $axa = a$.
4. A is reduced and $\operatorname{Spec} A$ is Hausdorff.
5. A is reduced and $\dim A = 0$.
6. For all $\mathfrak{p} \in \operatorname{Spec} A$, the local ring $A_{\mathfrak{p}}$ is a field.

We will denote by $a^{(-1)}$ the element x of 2., and by e_a the idempotent $aa^{(-1)}$. The set of elements of A which have a weak inverse in A will be denoted by $A^{(\times)}$.

Proof. 1 ⇒ 2 Take an element $a \in A$ and consider the exact sequence :

$$0 \longrightarrow (a) \longrightarrow A \longrightarrow A/(a) \longrightarrow 0$$

Since A is absolutely flat, $A/(a)$ is flat, and after applying $\otimes_A A/(a)$ to this sequence we get:

$$0 \longrightarrow (a)/(a^2) \longrightarrow A/(a) \longrightarrow A/(a) \longrightarrow 0$$

The arrow on the right is just the identity of $A/(a)$, so $(a)/(a^2) = 0$ and $(a) = (a^2)$.

2 ⇒ 3 From 2., we know that there is a $y \in A$ such that $a = a^2 y$. One can easily check that $x := yay$ verifies $xax = x$ and $axa = a$. Why is it unique ? Suppose there is a $z \in A$ satisfying the same conditions. Set $e = ax$, $f = az$. Then

$$(a) = (e) = (f)$$

and since e and f are idempotents generating the same ideal, they are equal, thus

$$(1 - e + a)(1 - e + x) = (1 - e + a)(1 - e + z) = 1$$

and by unicity of the inverse of $1 - e + a$, we have $x = z$.

3 ⇒ 4 First, A is reduced because if $a^n = 0$ then $a^{n-1} = aa^{(-1)} = 0$, so $a = 0$. Now take $\mathfrak{p} \neq \mathfrak{q} \in \operatorname{Spec} A$, so there is e.g. $f \in \mathfrak{p} \setminus \mathfrak{q}$. Then there is an idempotent $e = ff^{(-1)}$ such that $(f) = (e)$, so $\mathfrak{p} \in V(e) = D(1-e)$ and $\mathfrak{q} \in D(e)$.

4 ⇒ 5 If $\mathfrak{p} \subsetneq \mathfrak{q} \in \operatorname{Spec} A$ then $\mathfrak{q} \in V(p) = \overline{\{p\}}$ so they can't be separated by two open sets.

5 ⇒ 6 Since $\dim A = 0$, the only prime ideal contained in $\mathfrak{p}$ is $\mathfrak{p}$, and so the only prime ideal of $A_\mathfrak{p}$ is $\mathfrak{p}A_\mathfrak{p}$. But A is reduced, so is $A_\mathfrak{p}$ is as well and the intersection of its prime ideals is (0): $\mathfrak{p}A_\mathfrak{p} = (0)$, so $A_\mathfrak{p}$ is a field.

6 ⇒ 1 An A-module M is flat iff every $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$. This is true because $A_\mathfrak{p}$ is a field.

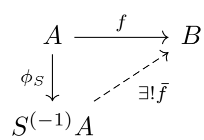
□

Corollary 1.1.

- Fields are absolutely flat.
- Any domain that is not a field is not absolutely flat, because it is at least one-dimensional.
- Any quotient or localisation of an absolutely flat ring is absolutely flat. Any product of absolutely flat rings is absolutely flat.
- If A is absolutely flat, then $\operatorname{Spec} A$ is compact and totally disconnected.

1.2 Construction of a universal absolutely flat ring

Consider a ring A and a subset $S \subseteq A$. One would like to construct an A-algebra $\phi_S : A \to S^{(-1)}A$ in which every element of S has a weak inverse, i.e. $\phi_S(S) \subseteq (S^{(-1)}A)^{(\times)}$. Mimicking localisation, this ring should satisfy the following universal property: for every ring homomorphism $f : A \to B$ such that $f(S) \subseteq B^{(\times)}$, there is a unique $\bar{f} : S^{(-1)}A \to B$ such that $f = \bar{f} \circ \phi_S$.



Note that this universal property immediately implies that $S^{(-1)}A$ is unique up to unique isomorphism, and that $A \to S^{(-1)}A$ is an epimorphism. In order to see what the ring $S^{(-1)}A$ looks like, let's start with the simple case where $S = \{s\}$, $s \in A$.

Proposition 1.2. $s^{(-1)}A \simeq A_s \times A/s$.

Proof. Let $f\colon A \to B$ be a ring homomorphism such that $f(s) \in B^{(\times)}$. Then $e = f(s)f(s)^{(-1)}$ is an idempotent, and

$$B \simeq B/e \times B/(1-e).$$

Notice that $f(s)$ is zero in B/e, and invertible in $B/(1-e)$. Hence there are unique morphisms $\phi : A/s \to B/e$, $\psi : A_s \to B/(1-e)$ compatible with f, and the morphism

$$\bar{f} = (\phi, \psi)\colon A/s \times A_s \to B = B/e \times B/(1-e)$$

extends f, and is the only one possible. □

Example 1.1. For an integer n, $n^{(-1)}\mathbb{Z} = \mathbb{Z}\left[\frac{1}{n}\right] \times \mathbb{Z}/n$.

In general, one can define $S^{(-1)}A$ by simply giving a weak inverse to every element of s.

Proposition 1.3. $S^{(-1)}A \simeq A\left[T_s | s \in S\right] / \left(sT_s s - s, T_s s T_s - T_s | s \in S\right)$.

Proof. We simply check that this construction satisfies the universal property of $S^{(-1)}A$. Consider a morphism $f\colon A \to B$ such that every $s \in S$ has a weak inverse. Then the only possibility of defining $\bar{f}$ is to set $\bar{f}(T_s) = f(s)^{(-1)}$, and this obviously works. □

Proposition 1.4. The map $\psi_S \colon \operatorname{Spec} S^{(-1)}A \to \operatorname{Spec} A$ is bijective.

Proof. We already know that $\phi_S : A \to S^{(-1)}A$ is an epimorphism, so ψ_S is a monomorphism of schemes: it is injective. Let us now prove that every $\mathfrak{p} \in \operatorname{Spec} A$ has a preimage. Denote by $k(\mathfrak{p})$ the residue field of A at $\mathfrak{p}$. Since $k(\mathfrak{p})^{(\times)} = k(\mathfrak{p})$, the universal property of $S^{(-1)}A$ gives us $\bar{\alpha} \colon S^{(-1)}A \to k(\mathfrak{p})$.

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & k(\mathfrak{p}) \\ {\scriptstyle \phi_S}\downarrow & \nearrow_{\bar{\alpha}} & \\ S^{(-1)}A & & \end{array}$$

Now by definition of α, we have

$$\mathfrak{p} = \ker \alpha = \phi_S^{-1}(\ker \beta)$$

and since $k(\mathfrak{p})$ is a field, $\ker \beta$ is a prime ideal of $S^{(-1)}A$, and $\psi_S(\ker \beta) = \mathfrak{p}$. ☐

Definition 1.2. The ring $A^{(-1)}A$ will be denoted by $T(A)$: it is called the universal absolutely flat ring associated with A. When X is the scheme $\operatorname{Spec} A$, the scheme $\operatorname{Spec} T(A)$ will be denoted by X^{cons}.

While it is clear that every element of A has a weak inverse in $T(A)$, we don't know yet whether every element of $T(A)$ does. This matter will be settled by the following proposition.

Proposition 1.5. The ring $T(A)$ is absolutely flat.

Proof. We are going to prove that for every prime ideal $\mathfrak{p}$ of $T(A)$, the local ring $T(A)_{\mathfrak{p}}$ is a field. More precisely, $A_{\mathfrak{p}}$ is the fraction field of $\phi(A)$ in $T(A)_{\mathfrak{p}}$, where $\phi : A \to T(A) \to T(A)_{\mathfrak{p}}$.

$\phi(A)$ is a domain First, let us note that $T(A)_{\mathfrak{p}}$ is local, hence connected: its only idempotents are 0 and 1. Now if $a, b \in A$ are such that

$$\phi(a)\phi(b) = 0$$

we may multiply by the weak inverses of $\phi(a)$ and $\phi(b)$ to get:

$$e_{\phi(a)} e_{\phi(b)} = 0.$$

Since $e_{\phi(a)}$ and $e_{\phi(b)}$ are idempotents, one of them must be 0, so $\phi(a) = e_{\phi(a)}\phi(a) = 0$ or $\phi(b) = e_{\phi(b)}\phi(b) = 0$.

$T(A)_{\mathfrak{p}}$ contains the fraction field K of $\phi(A)$ For every element $a \in A$, since $e_{\phi(a)} = \phi(a)\phi(a)^{(-1)}$ is idempotent, it is either 0 or 1. In the first case, $\phi(a) = e_{\phi(a)}\phi(a) = 0$; in the latter, $\phi(a)$ is invertible.

$T(A)_{\mathfrak{p}} = K$ Since $A \to T(a)$ is an epimorphism, $A \to T(A)_{\mathfrak{p}}$ is one as well, and so is the map $K \to T(a)_{\mathfrak{p}}$. But a ring epimorphism from a field to a ring is always an isomorphism, so $T(a)_{\mathfrak{p}} = K$. ☐

1.3 Absolutely flat schemes

There is a global notion of absolute flatness, defined by the local property satisfied by absolutely flat rings.

Definition 1.3. A scheme X is **absolutely flat** if at every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a field.

Remark 1.1. Unsurprisingly, this means that an affine scheme $X = \operatorname{Spec} A$ is absolutely flat if and only if A is absolutely flat.

Proposition 1.6. Let X be a scheme. If X is absolutely flat, then the ring of global sections $\Gamma(X, \mathcal{O}_X)$ is absolutely flat.

Proof. We are going to prove that, given $f \in A := \Gamma(X, \mathcal{O}_X)$, f^2 divides f. For every $x \in X$, the set $X_f := \{x \in X | f(x) \neq 0\}$ is clopen. It is obviously open, and is also closed because if $x \notin X_f$, $f(x)$ is invertible in $\mathcal{O}_{X,x}$ because $\mathcal{O}_{X,x}$ is a field, and so one can find an inverse of f in some open neighbourhood of x. This implies that the map:

$$A \longrightarrow \Gamma(X_f, \mathcal{O}_X) \times \Gamma(X \setminus X_f, \mathcal{O}_X)$$

is an isomorphism. By definition of X_f, f is invertible in $\Gamma(X_f, \mathcal{O}_X)$, so there is a $g \in A$ such that $gf^2 = f$ on X_f. Now since $f = 0$ on $X \setminus X_f$, the equality

$$f = gf^2$$

holds in A. □

Proposition 1.7. Let X be an absolutely flat scheme. The following are equivalent:

1. X is affine.
2. X is coherent.
3. X is compact and totally disconnected.

Proof. We already know that when X is affine, it is coherent, compact and totally disconnected. We will not use this result, the rest of the proof can be found in [Fer03]. □

2 The constructible topology

2.1 Constructible sets

Definition 2.1. A subset U of a topological space X is **retro-compact** if the inclusion map $U \hookrightarrow X$ is quasi-compact.

Definition 2.2. A **constructible** subset of a topological space X is a finite union of sets of the form $U \cap V^c$, where U and V are retro-compact open sets of X.

Remark 2.1. In some particular cases, constructible subsets can be described in a simpler way.

- If X is quasi-compact, then retro-compact is the same as quasi-compact.
- If X is a coherent (qcqs) scheme, the constructible subsets of X are exactly the quasi-compact sets of X.
- If X is a noetherian scheme, the constructible subsets of X are simply the finite unions of locally closed subsets of X.

Proposition 2.1. Finite unions, finite intersections and complements of constructible sets are constructible sets are constructible.

Definition 2.3. The **constructible topology** on a topological space X is the topology generated by constructible subsets of X.

2.2 The topology of X^{cons}

Let A be a ring, and $X = \operatorname{Spec} A$. We will now study the topology of $X^{\text{cons}} = \operatorname{Spec} T(A)$. Consider $\phi\colon X^{\text{cons}} \to X$. For $I \subseteq A$, we will denote by $D_A(I)$ and $V_A(I)$ the sets $\phi^{-1}(D(I))$ and $\phi^{-1}(V(I))$ respectively.

Proposition 2.2. X^{cons} is compact.

Proof. Since $T(A)$ is absolutely flat, X^{cons} is Hausdorff. Moreover, it is an affine scheme, so it is quasi-compact. □

Proposition 2.3. For all $a \in A$, the sets $V_A(a)$ and $D_A(a)$ are clopen in $X^{\text{cons}} = \operatorname{Spec} T(A)$.

Proof. Let $e = aa^{(-1)}$ be the idempotent associated with a in the absolutely flat ring $T(A)$. Then $V_A(a) = V(e) = D(1-e)$ is open and closed in X^{cons}. □

Corollary 2.1. For every finitely generated ideal $J \subseteq A$ and every $f \in A$, the set $V_A(J) \cap D_A(f)$ is clopen in X^{cons}.

The following lemma will help us characterise the topology of X^{cons}.

Lemma 2.1. Let τ, τ' be two topologies on a set X such that τ' is coarser than τ. If τ' is Hausdorff and τ is compact, then $\tau = \tau'$.

Proof. Consider the identity map $i : (X, \tau) \to (X, \tau')$. It is continuous because $\tau' \subseteq \tau$. We want to show that every closed set for τ is also closed for τ'. Let $C \subseteq X$ be a closed subset for the topology τ. Then since τ is compact, C is compact, so by continuity of i, C is compact for the Hausdorff topology τ', hence it is closed for τ'. □

Proposition 2.4. The topology of X^{cons} is generated by sets of the form $V_A(J) \cap D_A(f)$, where $J \subseteq A$ is a finitely generated ideal and $f \in A$.

Proof. The second topology is Hausdorff because if $\mathfrak{p} \neq \mathfrak{q} \in \operatorname{Spec} T(A)$, then $\mathfrak{p} \in V(f)$ and $\mathfrak{q} \in D(f)$. It is coarser than the topology of X^{cons} because the sets of the form $V_A(J) \cap D_A(f)$ are open in X^{cons}. Hence according the previous lemma, the two topologies are equal. □

Proposition 2.5. The constructible topology on $X = \operatorname{Spec} A$ is the Zariski topology of X^{cons}.

Proof. The proof is the same as the last one: the constructible topology is Hausdorff and coarser than the topology of X^{cons}. Indeed, since X is affine, a constructible set C is a finite union of $U_i \cap V_i^c$ where U_i, V_i are quasi-compact: we can write $U_i = D_A(a_1) \cup \cdots \cup D_A(a_r)$ and $V_i = (D_A(b_1) \cup \cdots \cup D_A(b_s))^c = V_A(b_1, \ldots, b_s)$. Now C is a finite union of sets of the form $V_A(J) \cap D_A(f)$, and it is open in X^{cons}. □

Proposition 2.6. A subset of $X = \operatorname{Spec} A$ is constructible if and only if it is clopen in X^{cons}.

Proof. As stated in the previous proof, a constructible subset of X is a finite union of $V_A(J) \cap D_A(f)$, so it is clopen in X^{cons}. Conversely, a clopen set C of X^{cons} is a union of finite intersections of $V_A(J_{ij}) \cap D_A(f_{ij})$. Since X^{cons} it compact and C is closed in X^{cons}, C is compact and this union can be chosen to be finite, so C is constructible in X. □

3 A proof of Chevalley's theorem

In this section, we are going to prove an affine version of Chevalley's theorem. The general statement and the reduction to the affine case can be found in chapter 10 of [Wed10].

Theorem 3.1. Let $f : X \to Y$ be a morphism of finite presentation between affine schemes. Then $f(X)$ is constructible in Y.

Proof. Write $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $B = A[t_1, \ldots, t_n]/J$ where J is finitely generated.

Reduction step 1 Since $X = V(J) \subseteq \mathbb{A}^n_Y$ is closed hence constructible, it is enough to prove that the image of a constructible subset of $\mathbb{A}^n_Y$ in $Y = \operatorname{Spec} A$ is constructible. If we prove it when $n = 1$, the result follows by induction.

Reduction step 2 As proved in 2.6, a constructible subset of $\mathbb{A}^1_Y$ is a union of sets of the form $V(J) \cap D(f)$, where $J \subseteq A[t]$ is finitely generated and $f \in A[t]$. Hence it suffices to prove that the image of the set $C := D(f) \cap V(J) = \operatorname{Spec}(A[t]/J)_f$ in Y is constructible.

Reduction to the absolutely flat case Suppose that we have proved this in the case where A is absolutely flat. Consider the scheme $Y^{\text{cons}} = \operatorname{Spec} T(A)$. We know that $(T(A)[t]/JT(A)[t])_f$ is absolutely flat. Let us call D its spectrum. We have the following commutative diagram, arising from the base change $Y^{\text{cons}} \to Y$:

$$\begin{array}{ccc} D & \xrightarrow{u} & C \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle r} \\ Y^{\text{cons}} & \xrightarrow{\phi} & Y \end{array}$$

We want to prove that $r(C)$ is constructible in Y. Since ϕ is surjective, u is as well, so

$$r(C) = \phi(s(D)).$$

By our assumption, $s(D)$ is constructible in Y^{cons} because $T(A)$ is absolutely flat, so it is clopen in Y^{cons} and $\phi(s(D))$ is constructible in Y. Hence $r(C)$ is constructible in Y.

Proof in the absolutely flat case We may now assume that A is absolutely flat. It suffices to show that for any $f \in A[t]/J$ where $J \subseteq A[t]$ is finitely generated, the image of $D(f) \subseteq V(J)$ in $Y = \operatorname{Spec} A$ is constructible. We are going to prove the following three facts, which will allow us to conclude the proof.

1. Let $I \subseteq A[t]$ be a finitely generated ideal. There exist orthogonal idempotents $e_1, \ldots, e_r \in A$ such that for every i, $(I + e_i A)/e_i A$ is either 0 or generated by a monic $g \in A[T]$.
2. The image of $D(f) \subseteq \mathbb{A}^1_Y$ in Y is open.
3. For all $f \in A[t]/g$ with g monic, $D(f) = \operatorname{Spec} A[t]/(g, e)$ for some $e \in A[t]$.

The first point implies that, using the Chinese remainder theorem, we can write

$$A[t]/J = \bigoplus_{i=1}^{r} (A/e_i)[t]/\left((J + e_i A)/e_i A\right)$$

and so $V(J)$ is the disjoint union of the spectra of these quotient rings. Since $\operatorname{Spec} A/e_i$ is clopen in Y, it suffices to show that the image of each of these spectra in $\operatorname{Spec} A/e_i$ is open. The ring A being absolutely flat, A/e_i is as well, so we may replace A/e_i by A and it will be enough to prove that the image of $\operatorname{Spec} A[t]/J$, with $J = 0$ or $J = (\text{monic } g)$, is open in $\operatorname{Spec} A$.

The second point deals with the case $J = 0$, and the third point with the case $J = (g)$. Indeed, the third point tells us that $D(f)$ is actually $\operatorname{Spec} A[t]/I$ with I finitely generated. Using the first fact, we know that its image in Y is open.

Proof of 1. For every $x \in Y$, the ideal $I \otimes k(x)$ of $k(x)[t]$ (where $k(x)$ is the residue field of Y at x) is either zero or generated by some monic polynomial f. If we write $I = (a_1 \ldots a_n)$, we have $s_1 \ldots s_n$ defined on some open neighbourhoods U_i of x such that $a_i = f s_i$, so on $U_x = \cap U_i$, the ideal $I \otimes \mathcal{O}_Y(U_x)$ is generated by f (or zero). Since A is absolutely flat, we can refine U_x to be $V(e_x)$ where e_x is an idempotent. Now Y is the union of all U_x, and this union can be chosen to be finite because Y is compact. It can even be made into a disjoint union: if $V(e)$ and $V(f)$ intersect, simply replace f with $f(1-e)$. Now we have $Y = \coprod V(e_i)$ for some idempotents $e_1 \ldots e_s$, with $I \otimes A/e_i = (I + e_i A)/e_i A$ being zero or generated by a monic polynomial.

Proof of 2. We proceed by induction on $\deg f$.
• If $\deg f = 0$, $f \in A$ so the image of $D(f)$ in Y is the open set $D_A(f)$.
• Induction step: call a the leading coefficient of f. If a is invertible, the image of $D(f)$ in Y is Y. If not, the image is the union of $D(a) \subseteq Y$ and of the image in $\operatorname{Spec} A/a = V(a)$ of $D(\bar{f})$, where $\bar{f}$ is the class of f in $(A/a)[t]$. Since $\deg \bar{f} < \deg f$, $D(\bar{f})$ is open in $V(a)$, which itself is open in Y because A is absolutely flat.

Proof of 3. Setting $B := A[t]/g$, we know that B is integral over A. Thus $\dim B \leqslant \dim A$, so $\dim B = 0$ because A is absolutely flat. This means that $B_{\text{red}} := B/\text{nilrad}B$ is zero-dimensional and reduced, hence absolutely flat. For $f \in B$, consider its class $\bar{f}$ in B_{red}. There is an idempotent $\bar{e} \in B_{\text{red}}$ such that

$$D(\bar{f}) = V(\bar{e}) = D(1 - \bar{e}) \subseteq \operatorname{Spec} B_{\text{red}}.$$

Thus in B, the elements $f - (1 - e)$ and $e(1 - e)$ are nilpotent, so

$$D(f) = D(1 - e) = V(e).$$

□

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