XVIII$_A$. Cohomological dimension : First results
Luc Illusie
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In this exposé we establish Gabber's bound on cohomological dimension stated in the introduction (in the comments on the proof of the finiteness theorem).

## 1. Bound in the strictly local case and applications

***Theorem 1.1***. — *Let $X$ be a strictly local, noetherian scheme of dimension $d > 0$, and let $\ell$ be a prime number invertible on $X$. Then, for any open subset $U$ of $X$, we have*

(1.1.1) $$\mathrm{cd}_\ell(U) \leq 2d - 1.$$

Recall that, for a scheme $S$, $\mathrm{cd}_\ell(S)$ ($\ell$-**cohomological dimension** of $S$) denotes the infimum of the integers $n$ such that for all $\ell$-torsion abelian sheaves $F$ on $S$, and all $i > n$, $H^i(S, F) = 0$.

***Corollary 1.2***. — *Let $X = \mathrm{Spec}\, A$ be as in 1.1, and assume $A$ is a domain, with fraction field $K$. Then*

(1.2.1) $$\mathrm{cd}_\ell(K) \leq 2d - 1.$$

Indeed, it suffices to show that if $F$ is a finitely generated $\mathbf{F}_\ell$-module over $\eta = \mathrm{Spec}\, K$, then $H^i(\eta, F) = 0$ for $i > 2d - 1$. But $\eta$ is a filtering projective limit of affine open subsets $U_\alpha$ of $X$, $F$ is induced from a locally constant constructible $\mathbf{F}_\ell$-sheaf $F_{\alpha_0}$ on $U_{\alpha_0}$, and $H^i(\eta, F) = \varinjlim H^i(U_\alpha, F_\alpha)$, where $F_\alpha = F_{\alpha_0}|U_\alpha$ for $\alpha \geq \alpha_0$ ([**SGA 4** VII 5.7]).

***Remark 1.3***. — (a) The proof shows that, given $X$ as in 1.1, with $X$ integral, then, if (1.1.1) holds for any affine open subset $U$, (1.2.1) holds, too.

(b) If $X$ is an integral noetherian scheme of dimension $d$, with generic point $\mathrm{Spec}\, K$, and $\ell$ is a prime number invertible on $X$, then $\mathrm{cd}_\ell(K) \geq d$ ([**SGA 4** X 2.5]). Gabber can prove that under the assumptions of 1.2 one has $\mathrm{cd}_\ell(K) = d$ (see XVIII$_B$).

***Corollary 1.4***. — *Let $Y$ be a noetherian scheme of finite dimension, $f : X \to Y$ a morphism of finite type, and $\ell$ a prime number invertible on $Y$. Then*

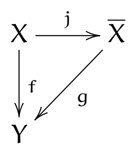
$$\mathrm{cd}_\ell(Rf_\star) < \infty,$$

*i.e. there exists an integer $N$ such that for any $\ell$-torsion abelian sheaf $F$ on $X$, $R^q f_\star F = 0$ for $q > N$.*

*Proof of 1.4*. We may assume $Y$ affine. Covering $X$ by finitely many open affine subsets $U_i$ ($0 \leq i \leq n$), and using the alternating Čech spectral sequence

$$E_1^{pq} = \bigoplus R^q(f|U_{i_0 \cdots i_p})_\star(F|U_{i_0 \cdots i_p}) \Rightarrow R^{p+q} f_\star F,$$

where $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$, we may assume $f$ separated. Repeating the procedure, we may assume $X$ affine. Choose an immersion $X \to \mathbf{P}^n_Y$, and replace $\mathbf{P}^n_Y$ by the scheme-theoretic closure of $X$. We get a commutative diagram




with $j$ open and $g$ projective of relative dimension $\leq n$. By the proper base change theorem we have $\mathrm{cd}_\ell(Rg_\star) \leq 2n$. By the Leray spectral sequence $R^p g_\star R^q j_\star F \Rightarrow R^{p+q} f_\star F$ it thus suffices to prove 1.4 for $j$, in other words, we may assume that $f$ is an open immersion. Let $d$ be the dimension of $Y$. Let $y$ be a geometric point of $Y$, and let

$U = Y_{(y)} \times_Y X$ be the corresponding open subset of the strictly local scheme $Y_{(y)}$ (of dimension $\leq d$, that we may assume to be $> 0$). Then

$$R^i f_\star(F)_y = H^i(U, F)$$

(where we still denote by $F$ its inverse image on $U$). The conclusion follows from 1.1.

***Remarks 1.5.*** — (a) Under the assumptions of 1.1, if $X$ is quasi-excellent and $U$ is affine, then by Gabber's affine Lefschetz theorem (exp. **XV**, 1.2.4) we have $cd_\ell(U) \leq d$. More generally, see exp. **XVII**, 3.2.1 for a proof of 1.1 for $X$ quasi-excellent.

(b) Gabber can show that, under the assumptions of 1.1, one has $cd_\ell(U) \geq d$ if $U$ is not empty and does not contain the closed point and that for each $n$ such that $d \leq n \leq 2d-1$, there exists a pair $(X, U)$ as in 1.1, with $U$ affine, such that $cd_\ell(U) = n$ (by (a), for $n > d$, $X$ is not quasi-excellent). These results are proved in $\mathrm{XVIII_B}$.

## 2. Proof of the main result

***Lemma 2.1.*** — *Let $X$ be as in 1.1, and let $x$ be the closed point of $X$. Then (1.1.1) holds for $U = X - \{x\}$.*

*Proof.* — It suffices to show that for any constructible $\mathbf{F}_\ell$-sheaf $F$ on $U$, $H^i(U, F) = 0$ for $i \geq 2d$. Let $\widehat{X}$ be the completion of $X$ at $\{x\}$ and set $\widehat{U} := \widehat{X} \times_X U = \widehat{X} - \{x\}$. Let $\widehat{F}$ be the inverse image of $F$ on $\widehat{U}$. By Gabber's formal base change theorem ([**Fujiwara, 1995**, 6.6.4]), the natural map

$$H^i(U, F) \to H^i(\widehat{U}, \widehat{F})$$

is an isomorphism for all $i$. Therefore we may assume $X$ complete, and in particular, excellent. Let $(f_1, \cdots, f_d)$ be a system of parameters of $X$, and let $U_i = X_{f_i}$, so that $U = \bigcup_{1 \leq i \leq d} U_i$. Consider the (alternate) Čech spectral sequence

$$E_1^{pq} = \bigoplus H^q(U_{i_0 \cdots i_p}, F) \Rightarrow H^{p+q}(U, F),$$

with $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ as above. By definition, $E_1^{pq} = 0$ for $p \geq d$. On the other hand, as $X$ is excellent, by Gabber's affine Lefschetz theorem (exp. **XV**, 1.2.4), we have $E_1^{pq} = 0$ for $q \geq d+1$. Therefore $E_1^{pq} = 0$ for $p + q \geq 2d$, hence $H^i(U, F) = 0$ for $i \geq 2d$. □

***Lemma 2.2.*** — *Let $X$ be a noetherian scheme of finite dimension, $Y$ a closed subset, $\ell$ a prime number invertible on $X$. Then, for any $\ell$-torsion sheaf $F$ on $X$,*

$$H^i_Y(X, F) = 0$$

*for*

$$i > \sup_{x \in Y}(cd_\ell(k(x)) + 2 \dim \mathscr{O}_{X,x}).$$

*In particular,*

$$cd_\ell(X) \leq \sup_{x \in X}(cd_\ell(k(x)) + 2 \dim \mathscr{O}_{X,x}).$$

*Proof.* — For $p \geq 0$, let $\Phi^p$ be the set of closed subsets of $Y$ of codimension $\geq p$ in $X$. We have $\Phi^p = \varnothing$ for $p > \dim(X)$. Consider the (biregular) coniveau spectral sequence of the filtration $(\Phi^p)$ (cf. [**Grothendieck, 1968**, 10.1]),

(2.2.1) $$E_1^{pq} = H^{p+q}_{\Phi^p/\Phi^{p+1}}(X, F) \Rightarrow H^{p+q}_Y(X, F).$$

We have

$$E_1^{pq} = \bigoplus_{x \in Y^{(p)}} H^{p+q}_{\{x\}}(X_x, F|X_x),$$

where $Y^{(p)}$ denotes the set of points of $Y$ of codimension $p$ in $X$, and $X_x = \operatorname{Spec} \mathscr{O}_{X,x}$. For $x \in Y^{(p)}$ (i.e. $\dim \mathscr{O}_{X,x} = p$), let $\overline{x}$ be a geometric point above $x$. Consider the diagram

$$\begin{array}{ccccc} \{\overline{x}\} & \xrightarrow{i_{\overline{x}}} & X_{(\overline{x})} & \xleftarrow{\overline{j}} & \overline{U}\,, \\ \downarrow & & \downarrow & & \downarrow \\ \{x\} & \xrightarrow{i_x} & X_x & \xleftarrow{j} & U \end{array}$$

where $U = X_x - \{x\}$, $\overline{U} = X_{(\overline{x})} - \{\overline{x}\}$. We have

$$R\Gamma_{\{x\}}(X_x, F|X_x) = R\Gamma(\{x\}, Ri_x^!(F|X_x)).$$

The stalk of $Ri^!_x(F|X_x)$ at $\overline{x}$ is

$$Ri^!_x(F|X_x)_{\overline{x}} = Ri^!_{\overline{x}}(F|X_{(\overline{x})}),$$

as $(Rj_\star(F|U))_{\overline{x}} = R\overline{j}_\star(F|\overline{U})_{\overline{x}}$. We thus have a spectral sequence

$$E_2^{rs} = H^r(k(x), R^s i^!_{\overline{x}}(F|X_{(\overline{x})})) \Rightarrow H^{r+s}_{\{x\}}(X_x, F|X_x), \tag{2.2.2}$$

It suffices to show that, in the initial term of (2.2.1),

$$H^{p+q}_{\{x\}}(X_x, F|X_x) = 0 \tag{2.2.3}$$

for $p+q > cd_\ell(k(x)) + 2p$. If $p = 0$, then $Ri^!_{\overline{x}}(F|X_{(\overline{x})}) = F_{\overline{x}}$, and, in (2.2.2), $E_2^{rs} = 0$ for $s > 0$, $E_2^{r0} = 0$ for $r > cd_\ell(k(x))$, so (2.2.3) is true in this case. Assume $p > 0$. We have

$$R^s i^!_{\overline{x}} F = H^{s-1}(\overline{U}, F|\overline{U}) \tag{2.2.4}$$

for $s \geq 2$, where, as above, $\overline{U} = X_{(\overline{x})} - \{\overline{x}\}$. By 2.1, $H^{s-1}(\overline{U}, F|\overline{U}) = 0$ for $s - 1 \geq 2p$, hence, by (2.2.4), $R^s i^!_{\overline{x}}(F|X_{(\overline{x})}) = 0$ and $E_2^{rs} = 0$ for $s \geq 2p+1$. If $r + s \geq cd_\ell(k(x)) + 2p + 1$ and $s \leq 2p$, then $r > cd_\ell(k(x))$, hence $E_2^{rs} = 0$ as well. Therefore, by (2.2.2), (2.2.3) holds, which finishes the proof. □

*Proof of 1.1.* We prove 1.1 by induction on $d$. For $n \geq 0$ consider the assertion

$G_n$ : *For every strictly local, noetherian scheme $X$ of dimension $n$, all open subsets $U$ of $X$ and any prime number $\ell$ invertible on $X$, we have* $cd_\ell(U) \leq \sup(0, 2n-1)$.

Let $d > 0$. Assume $G_n$ holds for $n < d$, and let us prove $G_d$. Let $X$ be as in 1.1. If $(X_i)_{1 \leq i \leq r}$ are the reduced irreducible components of $X$ and $U_i = U \times_X X_i$, we have $cd_\ell(U) \leq \sup(cd_\ell(U_i))$, hence we may assume $X$ integral. Let $x$ be the closed point of $X$, and $U = X - \{x\}$ the punctured spectrum. Let $j : V \to U$ be a nonempty open subset of $U$, and $F$ be a constructible $\mathbf{F}_\ell$-sheaf on $V$. As $F = j^\star j_! F$, by 2.1 it suffices to show that, for any constructible $\mathbf{F}_\ell$-sheaf $L$ on $U$, the restriction map

$$H^i(U, L) \to H^i(V, j^\star L) \tag{$\ast$}$$

is an isomorphism for $i \geq 2d$. Let $Y = U - V$. Consider the exact sequence

$$H^i_Y(U, L) \to H^i(U, L) \to H^i(V, j^\star L) \to H^{i+1}_Y(U, L).$$

By 2.2, we have $H^i_Y(U, L) = 0$ for $i > \sup_{y \in Y}(cd_\ell(k(y)) + 2\dim \mathcal{O}_{X,y})$. For $y \in Y$, denote by $Z$ the closed, integral subscheme of $X$ defined by the closure of $\{y\}$ *in* $X$. As $X$ is integral and $V$ nonempty, $Z$ is a strictly local scheme of dimension $n < d$, with generic point $y$. By 1.3 (a) and $G_n$ (inductive assumption), we have $cd_\ell(k(y)) \leq 2n-1$. We have $2n - 1 + 2\dim \mathcal{O}_{X,y} \leq 2d - 1$. Hence, for $i \geq 2d$, $H^i_Y(U, L) = H^{i+1}_Y(U, L) = 0$, and $(\ast)$ is an isomorphism, which finishes the proof.

***Remark 2.3.*** — Gabber has an alternate proof of 1.1, based on the theory of Zariski-Riemann spaces. By 2.2, it suffices to show 1.2. Here is a sketch, pasted from an e-mail of Gabber to Illusie of 2007, Aug. 15 :

"For $Y \to X$ proper birational with special fiber $Y_0$, consider $i : Y_0 \to Y$ and $j : \eta \to Y$, $\eta$ the generic point. We have by proper base change a spectral sequence

$$H^p(Y_0, i^\star R^q j_\star F) \to H^{p+q}(\eta, F)$$

for $F$ an $\ell$-torsion Galois module. We take the direct limit and get a spectral sequence involving cohomologies on the étale topos of $ZRS_0$ defined as the limit of étale topoi of $Y_0$ or viewing $ZRS_0$ as a locally ringed topos and applying a universal construction in the book of M. Hakim. The limit of the $R^q j_\star F$ is $R^q(\eta \to ZRS)_\star F$. By a classical result of Abhyankar, also proved in Appendix 2 of the book of Zariski-Samuel Vol. II, if $R$ is a noetherian local domain of dimension $d$ and $V$ a valuation ring of $\mathrm{Frac}(R)$ dominating $R$, the sum of the rational rank and the residue transcendence degree is at most $d$. For a strictly henselian valuation ring $V$ with residue characteristic exponent $p$ and value group $\Gamma$, the absolute Galois group of $\mathrm{Frac}(V)$ is an extension of the tame part (product for $\ell$ prime not equal to $p$ of $\mathrm{Hom}(\Gamma, \mathbf{Z}_\ell(1))$) by a $p$-group, so the $\ell$-cohomological dimension is the dimension of $\Gamma$ tensored with the prime field $\mathbf{F}_\ell$, which is at most the dimension of $\Gamma$ tensored with the rationals. If $A$ is an $\ell$-torsion sheaf on the étale topos of $ZRS_0$, let $\delta(A)$ be the sup of transcendence degrees of points where the stalk is non-zero. I claim that $H^n(ZRS_0, A)$ vanishes for $n > 2\delta(A)$. One reduces it to the finite type case (passage to the limit [**SGA 4** VI 8.7.4]) using that the $\delta$ of the direct image of $A$ to $Y_0$ is at most $\delta(A)$. In $Y_0$ the transcendence degrees over the closed point of $X$ are at most $d-1$ by the dimension inequality. Summing up, for the limit spectral sequence the $q$-th direct image sheaf restricted to the special fiber has $\delta$ at most $\min(d-1, d-q)$, giving vanishing for certain $E_2^{p,q}$ and the result."

## Références

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LUC ILLUSIE